Caltech

Safety of Sampled-Data Systems with Control Barrier Functions via Approximate Discrete Time Models

Andrew J. Taylor Victor D. Dorobantu Ryan K. Cosner
Yisong Yue Aaron D. Ames

Computing and Mathematical Sciences
California Institute of Technology

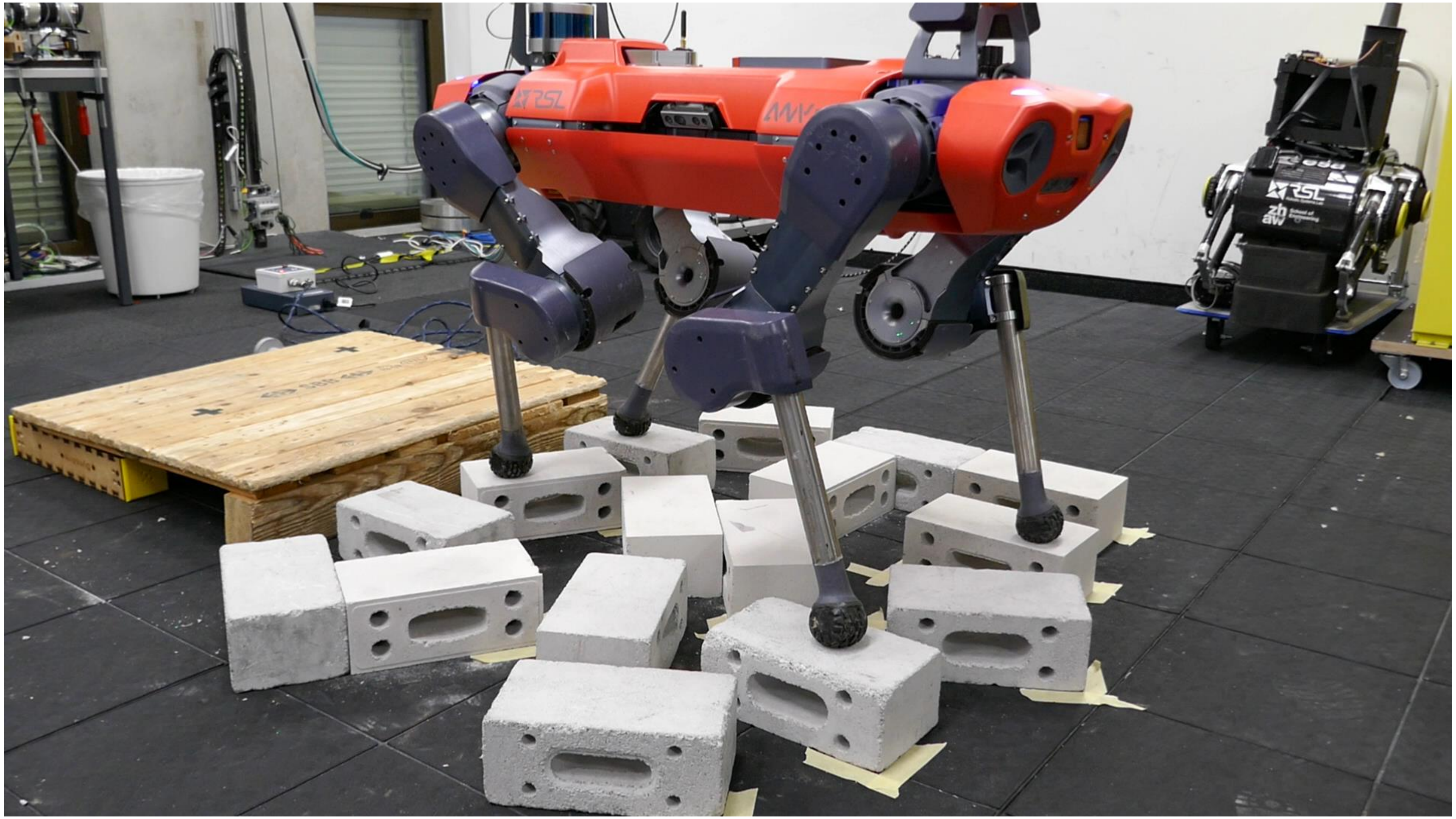
December 9th, 2022
Control & Decision Conference (CDC) 2022

Control in the real world is hard

Caltech

But: Pretty when it works…

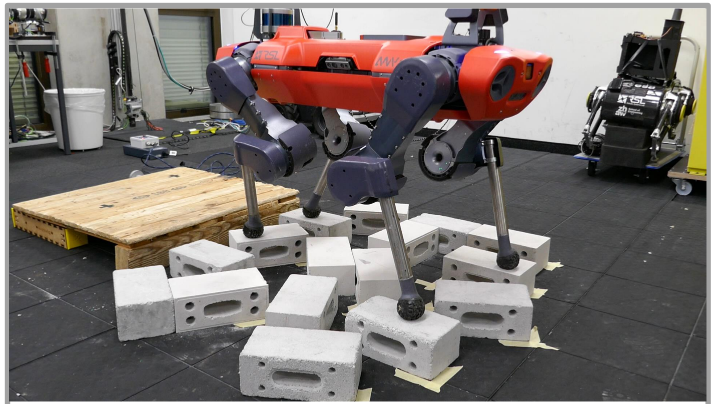
Caltech



[1] R. Grandia, **A. J. Taylor**, M. Hutter, A. D. Ames, "Multi-Layered Safety for Legged Robotics via Control Barrier Functions and Model Predictive Control", 2020.

Claim: Need to Bridge the Gap

Caltech



$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\text{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t. } \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x}))$$

Theorems & Proofs

Bridge the Gap

$$h(\mathbf{x}_{i+1}) - h(\mathbf{x}_i) \geq -\alpha(h(\mathbf{x}_i))$$

Experimental Realization

Contributions

Caltech

- Framework for achieving safety of sampled-data systems via **Control Barrier Functions (CBFs)** and **approximate discrete time models**
- Definition of **practical safety** analogous to that of practical stability for sampled-data systems
- Analysis of relationship between a CBF and an approximate discrete time model that yields **convex optimization-based** controllers

System Dynamics

Caltech

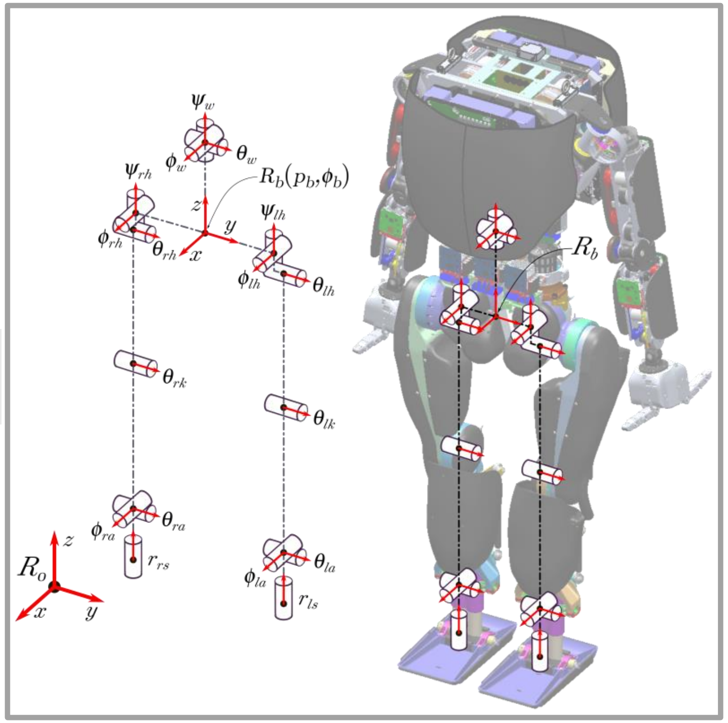
Equations of Motion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \qquad \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^{n \times m}$$

Mathematical Model



System Model

System Dynamics

Caltech

Equations of Motion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

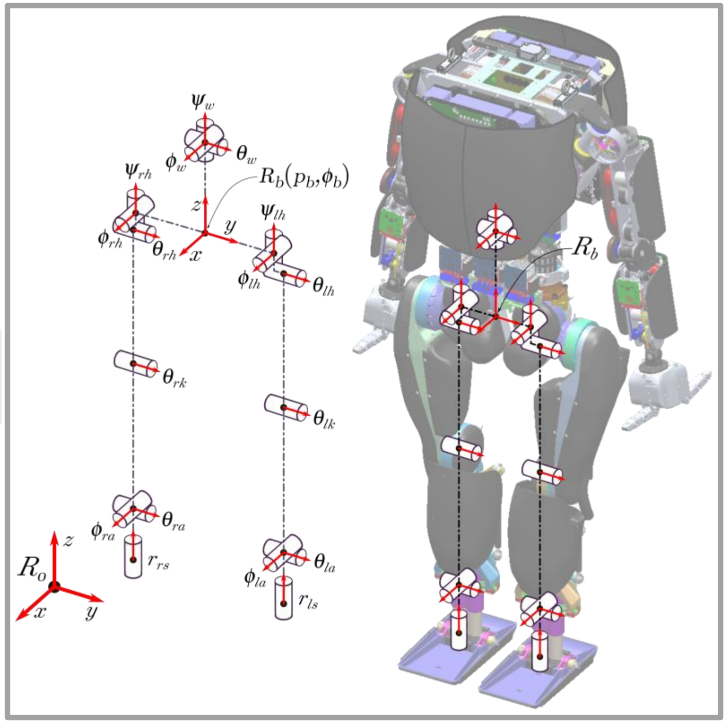
$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \qquad \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$$

Assumptions

$\mathbf{f}$, $\mathbf{g}$ locally Lipschitz continuous

Mathematical Model



System Model

System Dynamics

Caltech

Equations of Motion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$
$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$
$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \qquad \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$$

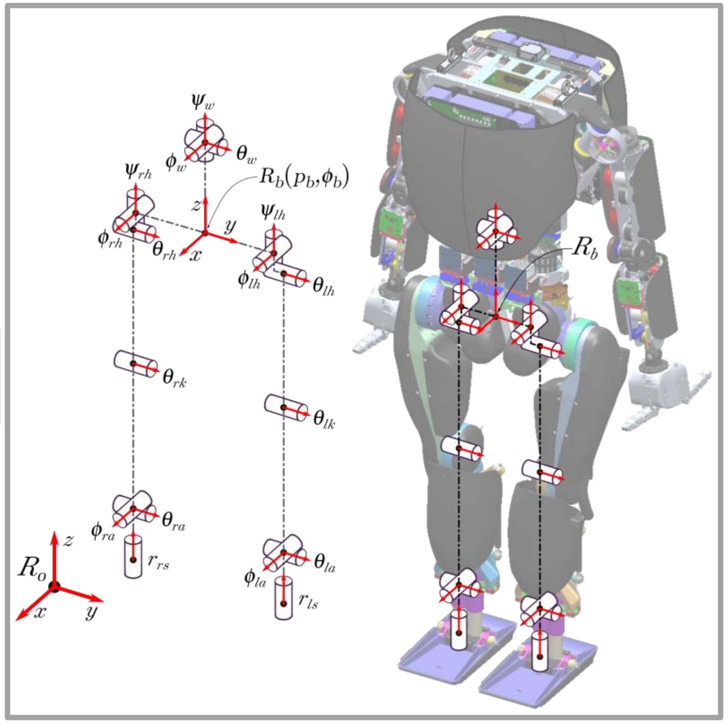
Assumptions

$\mathbf{f}$, $\mathbf{g}$ locally Lipschitz continuous

Closed-Loop Solutions

$$\mathbf{k}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$$
$$\mathbf{x}_0 \in \mathbb{R}^n \qquad \boldsymbol{\varphi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$$
$$\dot{\boldsymbol{\varphi}}(t) = \mathbf{f}(\boldsymbol{\varphi}(t)) + \mathbf{g}(\boldsymbol{\varphi}(t))\mathbf{k}(\boldsymbol{\varphi}(t))$$
$$\boldsymbol{\varphi}(0) = \mathbf{x}_0$$

Mathematical Model



System Model

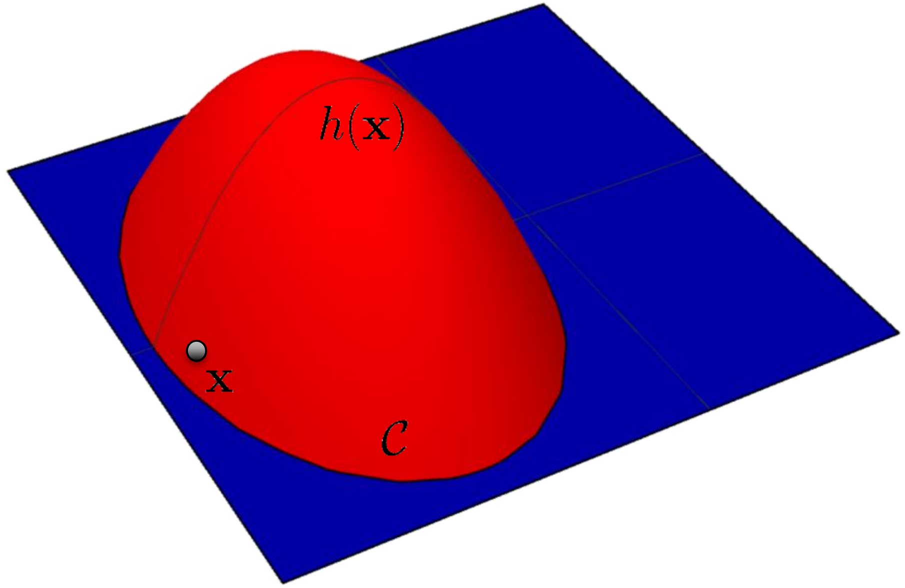
Barrier Functions (BFs)

Caltech

Safe Set

$$h : \mathbb{R}^n \to \mathbb{R}$$

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h(\mathbf{x}) \geq 0\}$$



Barrier Functions (BFs)

Caltech

Safe Set

$$h : \mathbb{R}^n \to \mathbb{R}$$

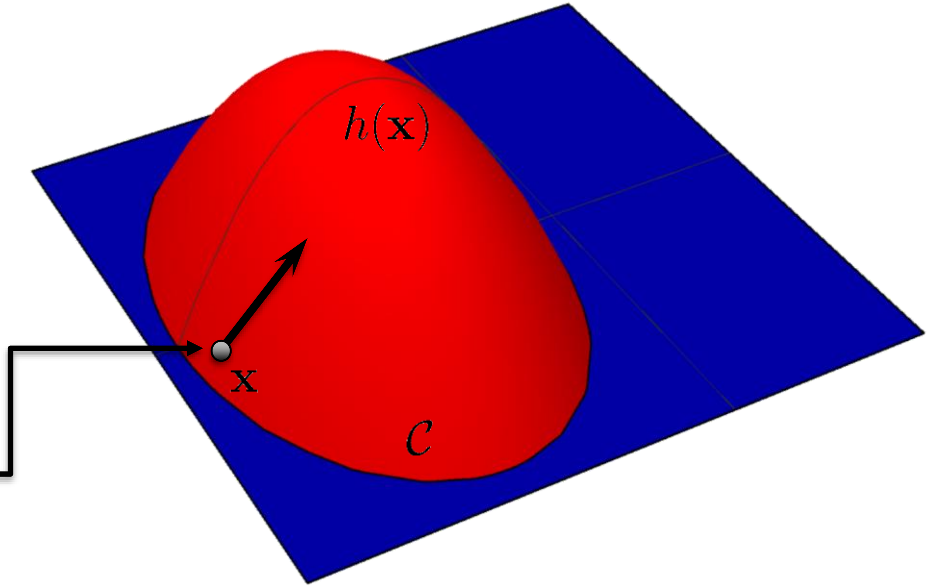
$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h(\mathbf{x}) \geq 0\}$$

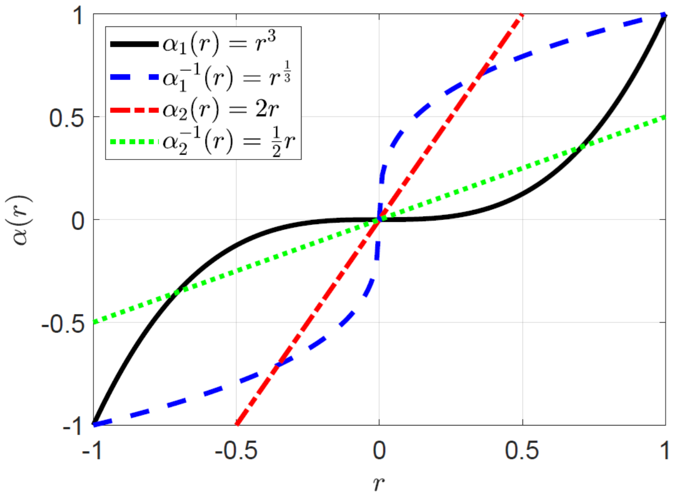
Barrier Function [2]

$$\dot{h}(\mathbf{x}, \mathbf{k}(\mathbf{x})) \geq -\alpha(h(\mathbf{x})) \ \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

$$\dot{h}(\mathbf{x}, \mathbf{u}) = \underbrace{\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}(\mathbf{x})}_{L_{\mathbf{f}}h(\mathbf{x})} + \underbrace{\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{g}(\mathbf{x})}_{L_{\mathbf{g}}h(\mathbf{x})}\mathbf{u}$$

$$\alpha \in \mathcal{K}_\infty^{\mathrm{e}}$$





[2] A. Ames, X. Xu, J. Grizzle, P. Tabuada, "Control Barrier Function Based Quadratic Programs for Safety Critical Systems", 2017.

Barrier Functions (BFs)

Caltech

Safe Set

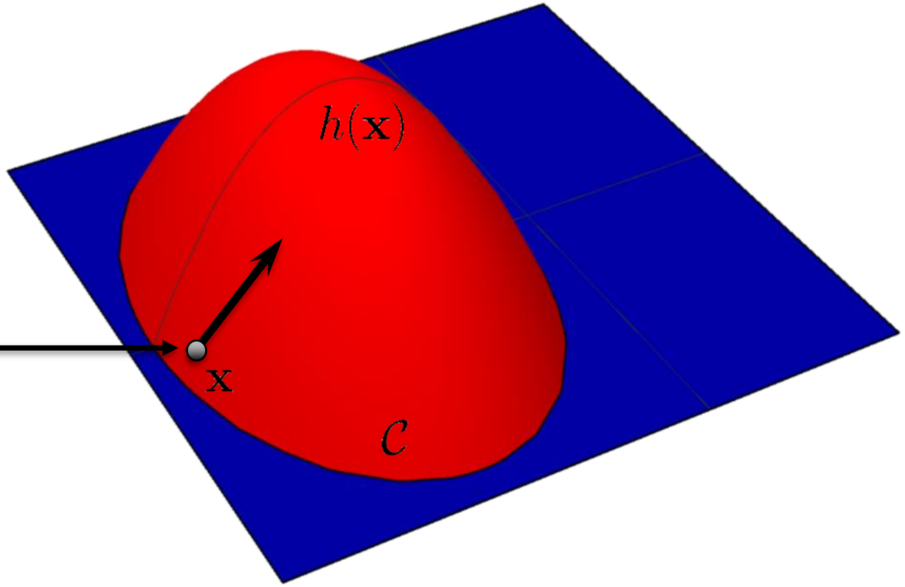
$$h : \mathbb{R}^n \to \mathbb{R}$$
$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h(\mathbf{x}) \geq 0\}$$

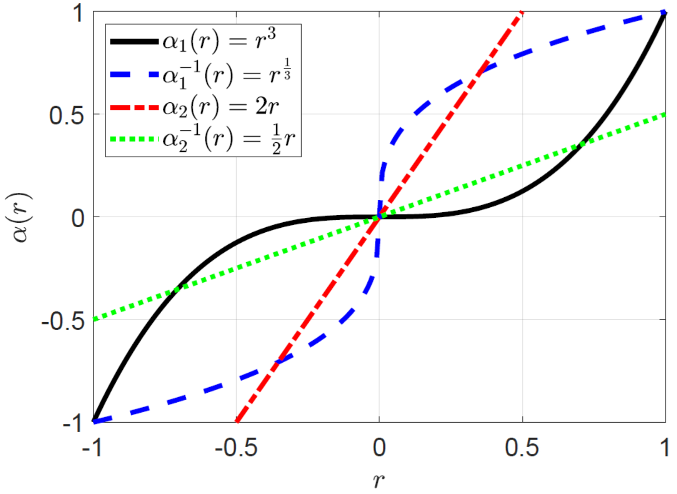
Barrier Function [2]

$$\dot{h}(\mathbf{x}, \mathbf{k}(\mathbf{x})) \geq -\alpha(h(\mathbf{x})) \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$
$$\dot{h}(\mathbf{x}, \mathbf{u}) = \underbrace{\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}(\mathbf{x})}_{L_{\mathbf{f}}h(\mathbf{x})} + \underbrace{\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{g}(\mathbf{x})}_{L_{\mathbf{g}}h(\mathbf{x})}\mathbf{u}$$
$$\alpha \in \mathcal{K}_\infty^{\mathrm{e}}$$

Safety [2]

$$\dot{h}(\mathbf{x}, \mathbf{k}(\mathbf{x})) \geq -\alpha(h(\mathbf{x})) \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$
$$\implies \mathcal{C} \text{ is forward invariant}$$





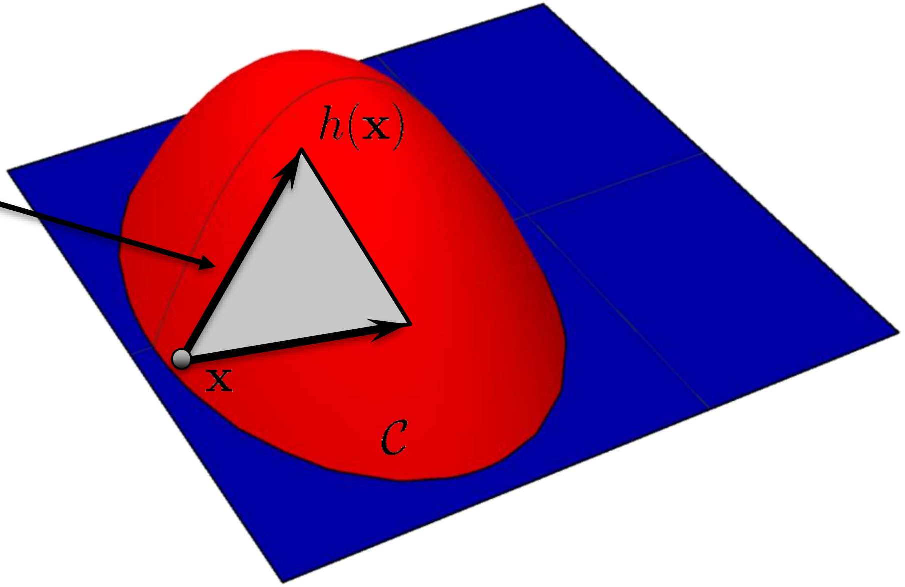
[2] A. Ames, X. Xu, J. Grizzle, P. Tabuada, "Control Barrier Function Based Quadratic Programs for Safety Critical Systems", 2017.

Control Barrier Functions (CBFs)

Caltech

Control Barrier Function [2]

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}(\mathbf{x}, \mathbf{u}) > -\alpha(h(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^n$$



[2] A. Ames, X. Xu, J. Grizzle, P. Tabuada, "Control Barrier Function Based Quadratic Programs for Safety Critical Systems", 2017.

Control Barrier Functions (CBFs)

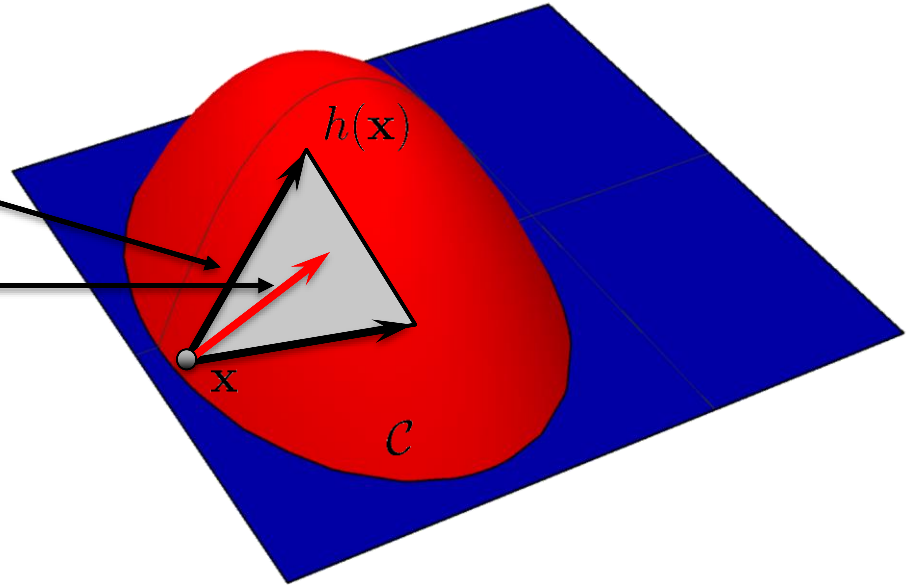
Caltech

Control Barrier Function [2]

$$\sup_{\mathbf{u}\in\mathbb{R}^m} \dot{h}(\mathbf{x},\mathbf{u}) > -\alpha(h(\mathbf{x})) \text{ for all } \mathbf{x}\in\mathbb{R}^n$$

CBF Quadratic Program [2]

$$\begin{aligned}\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} & \ \|\mathbf{u}-\mathbf{k}_{\mathrm{nom}}(\mathbf{x})\|_2^2 \\ \text{s.t.} & \ \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x}))\end{aligned}$$



[2] A. Ames, X. Xu, J. Grizzle, P. Tabuada, "Control Barrier Function Based Quadratic Programs for Safety Critical Systems", 2017.

Control Barrier Functions (CBFs)

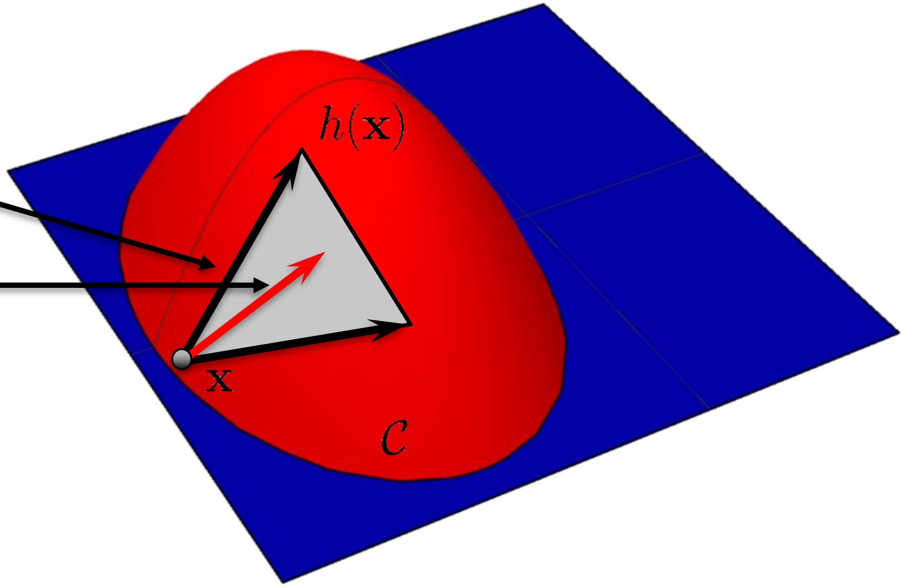
Caltech

Control Barrier Function [2]

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}(\mathbf{x}, \mathbf{u}) > -\alpha(h(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

CBF Quadratic Program [2]

$$\begin{aligned} \mathbf{k}(\mathbf{x}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} & \ \|\mathbf{u} - \mathbf{k}_{\mathrm{nom}}(\mathbf{x})\|_2^2 \\ \text{s.t.} & \ \dot{h}(\mathbf{x}, \mathbf{u}) \geq -\alpha(h(\mathbf{x})) \end{aligned}$$

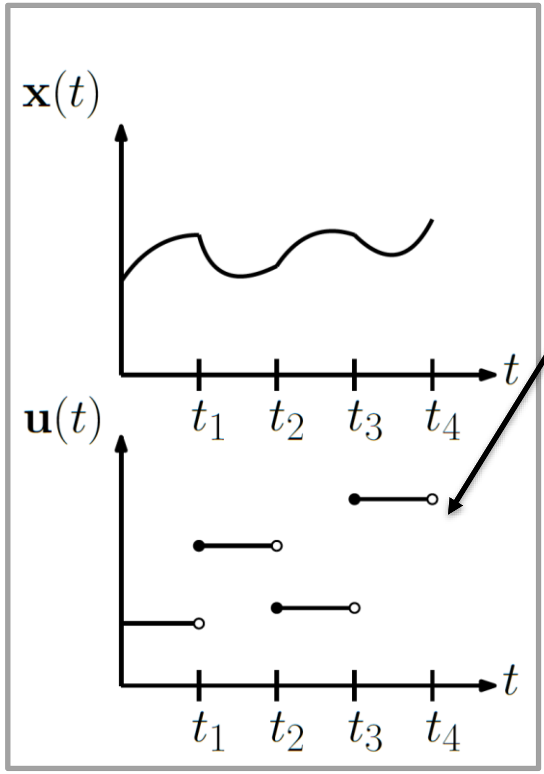


How are these controllers implemented?

[2] A. Ames, X. Xu, J. Grizzle, P. Tabuada, "Control Barrier Function Based Quadratic Programs for Safety Critical Systems", 2017.

Sampled-Data Control

Caltech



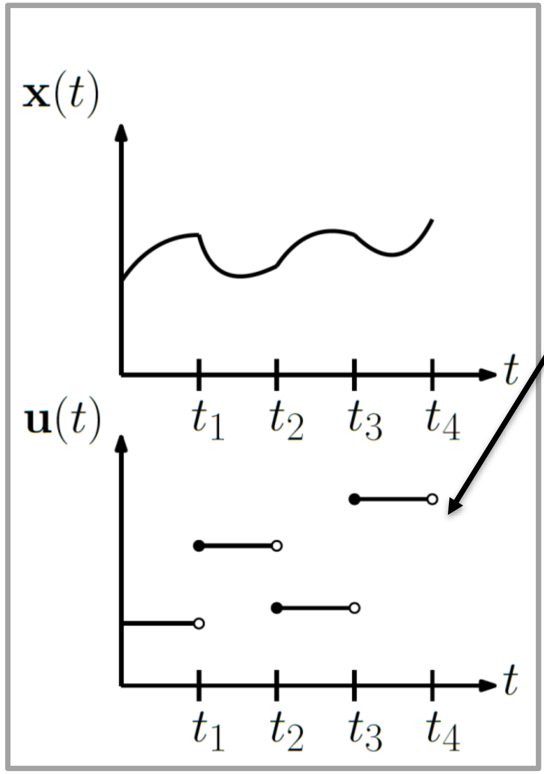
Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$

$$t_{i+1} - t_i = T$$

Sampled-Data Control

Caltech



Sample-and-Hold

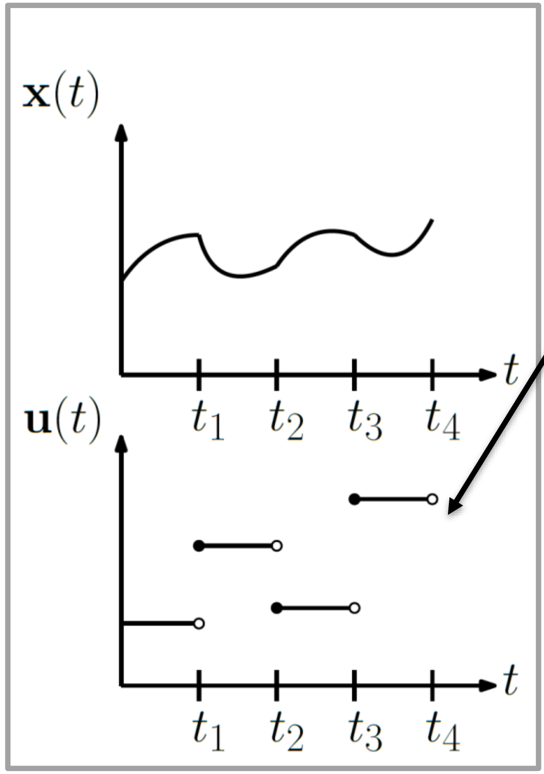
$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$

$$t_{i+1} - t_i = T$$

Recent Work

[3] A. Ghaffari, I. Abel, D. Ricketts, S. Lerner, M. Krstić, "Safety Verification Using Barrier Certificates with Application to Double Integrator with Input Saturation and Zero-Order Hold", 2018.
[4] W. S. Cortez, D. Oetomo, C. Manzie, P. Choong, "Control Barrier Functions for Mechanical Systems: Theory and Application to Robotic Grasping", 2021.
[5] J. Breeden, K. Garg, D. Panagou, "Control Barrier Functions in Sampled-Data Systems", 2021.
[6] J. Usevitch, D. Panagou, "Adversarial Resilience for Sampled-Data Systems Using Control Barrier Function Methods", 2021.
[7] L. Niu, H. Zhang, A. Clark, "Safety-Critical Control Synthesis for Unknown Sampled-Data Systems via Control Barrier Functions", 2021.

Sampled-Data Control



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Recent Work

[3] A. Ghaffari, I. Abel, D. Ricketts, S. Lerner, M. Krstić, "Safety Verification Using Barrier Certificates with Application to Double Integrator with Input Saturation and Zero-Order Hold", 2018.
[4] W. S. Cortez, D. Oetomo, C. Manzie, P. Choong, "Control Barrier Functions for Mechanical Systems: Theory and Application to Robotic Grasping", 2021.
[5] J. Breeden, K. Garg, D. Panagou, "Control Barrier Functions in Sampled-Data Systems", 2021.
[6] J. Usevitch, D. Panagou, "Adversarial Resilience for Sampled-Data Systems Using Control Barrier Function Methods", 2021.
[7] L. Niu, H. Zhang, A. Clark, "Safety-Critical Control Synthesis for Unknown Sampled-Data Systems via Control Barrier Functions", 2021.

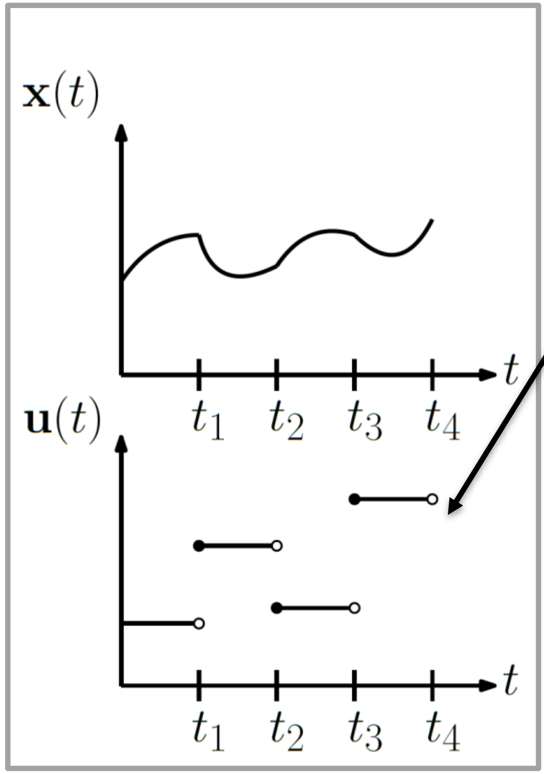
Emulation Approach[8]

$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\mathrm{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t. } \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x})) + \phi(T)$$

[8] S. Monaco, D. Normand-Cyrot, "Advanced Tools for Nonlinear Sampled-Data Systems' Analysis and Control", 2007.

Sampled-Data Control

Caltech



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Recent Work

[3] A. Ghaffari, I. Abel, D. Ricketts, S. Lerner, M. Krstić, "Safety Verification Using Barrier Certificates with Application to Double Integrator with Input Saturation and Zero-Order Hold", 2018.
[4] W. S. Cortez, D. Oetomo, C. Manzie, P. Choong, "Control Barrier Functions for Mechanical Systems: Theory and Application to Robotic Grasping", 2021.
[5] J. Breeden, K. Garg, D. Panagou, "Control Barrier Functions in Sampled-Data Systems", 2021.
[6] J. Usevitch, D. Panagou, "Adversarial Resilience for Sampled-Data Systems Using Control Barrier Function Methods", 2021.
[7] L. Niu, H. Zhang, A. Clark, "Safety-Critical Control Synthesis for Unknown Sampled-Data Systems via Control Barrier Functions", 2021.

Emulation Approach[8]

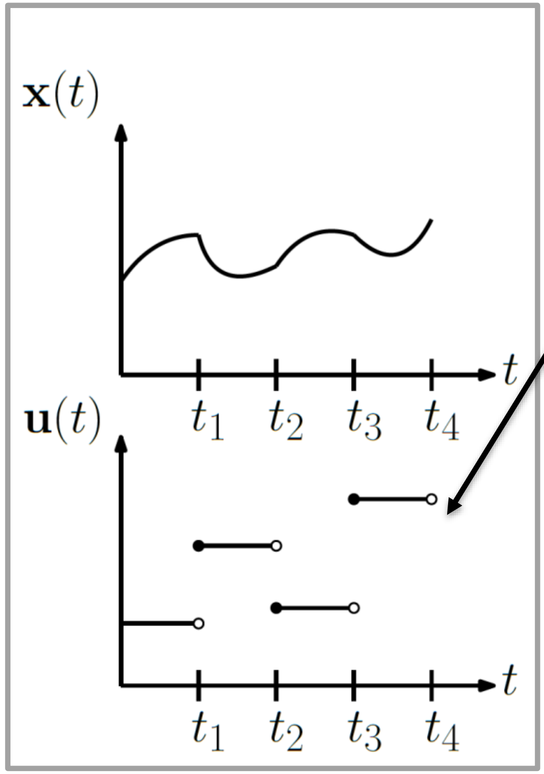
$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\text{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t. } \dot{h}(\mathbf{x}, \mathbf{u}) \geq -\alpha(h(\mathbf{x})) + \phi(T)$$

Strong theoretical guarantees on inter-sample behavior

[8] S. Monaco, D. Normand-Cyrot, "Advanced Tools for Nonlinear Sampled-Data Systems' Analysis and Control", 2007.

Sampled-Data Control

Caltech



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Recent Work

[3] A. Ghaffari, I. Abel, D. Ricketts, S. Lerner, M. Krstić, "Safety Verification Using Barrier Certificates with Application to Double Integrator with Input Saturation and Zero-Order Hold", 2018.
[4] W. S. Cortez, D. Oetomo, C. Manzie, P. Choong, "Control Barrier Functions for Mechanical Systems: Theory and Application to Robotic Grasping", 2021.
[5] J. Breeden, K. Garg, D. Panagou, "Control Barrier Functions in Sampled-Data Systems", 2021.
[6] J. Usevitch, D. Panagou, "Adversarial Resilience for Sampled-Data Systems Using Control Barrier Function Methods", 2021.
[7] L. Niu, H. Zhang, A. Clark, "Safety-Critical Control Synthesis for Unknown Sampled-Data Systems via Control Barrier Functions", 2021.

Emulation Approach[8]

$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\mathrm{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t. } \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x})) + \phi(T)$$

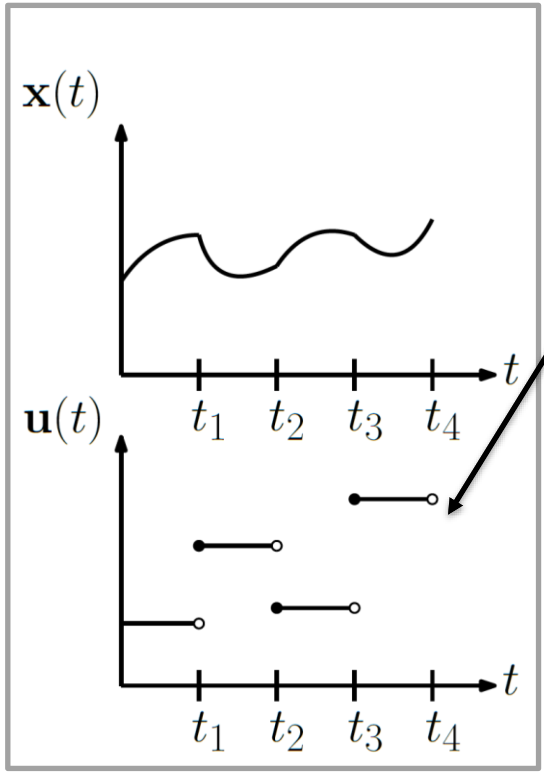
$$\phi(T) = a(e^{bT} - 1)$$
a, b Lipschitz based

Strong theoretical guarantees on inter-sample behavior

[8] S. Monaco, D. Normand-Cyrot, "Advanced Tools for Nonlinear Sampled-Data Systems' Analysis and Control", 2007.

Sampled-Data Control

Caltech



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Recent Work

[3] A. Ghaffari, I. Abel, D. Ricketts, S. Lerner, M. Krstić, "Safety Verification Using Barrier Certificates with Application to Double Integrator with Input Saturation and Zero-Order Hold", 2018.
[4] W. S. Cortez, D. Oetomo, C. Manzie, P. Choong, "Control Barrier Functions for Mechanical Systems: Theory and Application to Robotic Grasping", 2021.
[5] J. Breeden, K. Garg, D. Panagou, "Control Barrier Functions in Sampled-Data Systems", 2021.
[6] J. Usevitch, D. Panagou, "Adversarial Resilience for Sampled-Data Systems Using Control Barrier Function Methods", 2021.
[7] L. Niu, H. Zhang, A. Clark, "Safety-Critical Control Synthesis for Unknown Sampled-Data Systems via Control Barrier Functions", 2021.

$$\phi(T) = a(e^{bT} - 1)$$
a, b Lipschitz based

Emulation Approach[8]

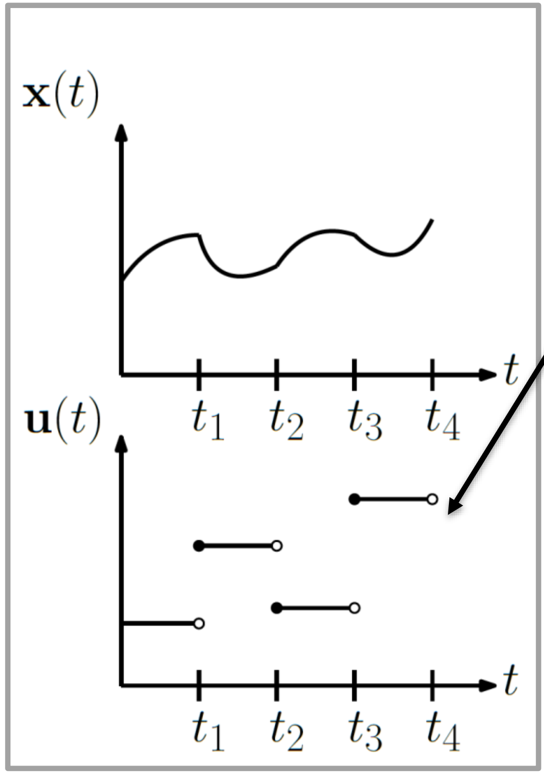
$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\text{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t. } \dot{h}(\mathbf{x}, \mathbf{u}) \geq -\alpha(h(\mathbf{x})) + \phi(T)$$

Strong theoretical guarantees on inter-sample behavior

[8] S. Monaco, D. Normand-Cyrot, "Advanced Tools for Nonlinear Sampled-Data Systems' Analysis and Control", 2007.

Sampled-Data Control

Caltech

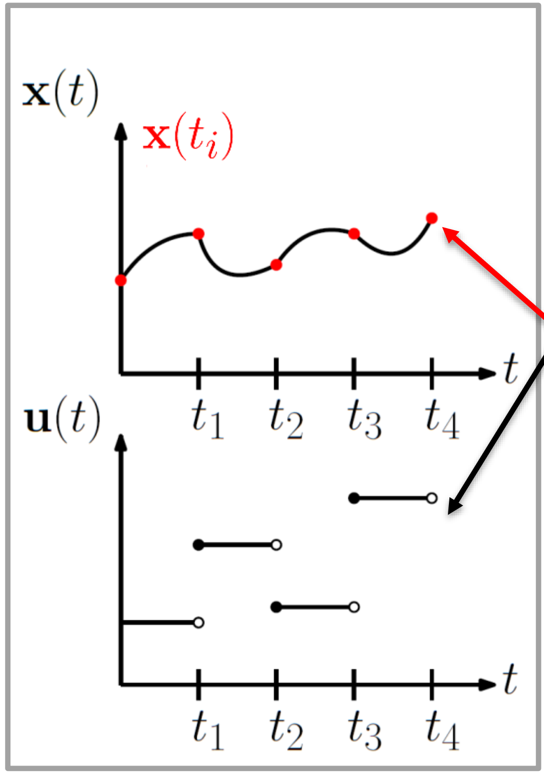


Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Sampled-Data Control

Caltech



Sample-and-Hold

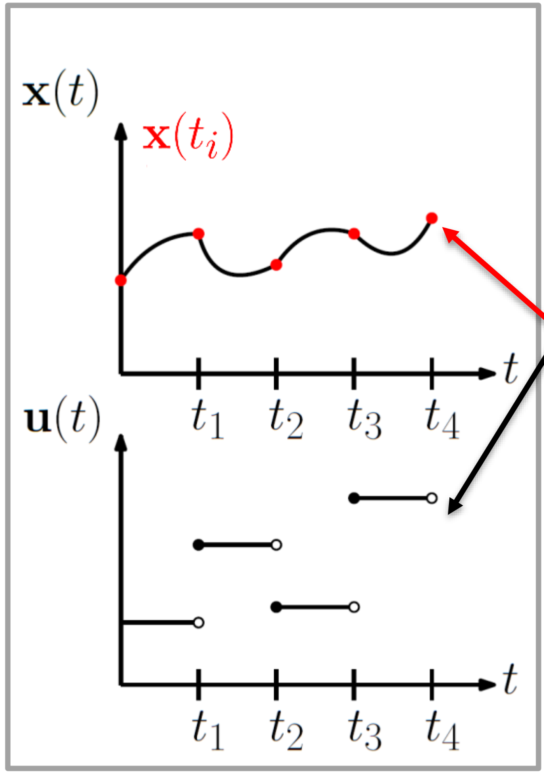
$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Discrete State Transitions

$$\mathbf{F}_T^e(\mathbf{x},\mathbf{u}) = \mathbf{x} + \int_0^T [\mathbf{f}(\varphi(\tau)) + \mathbf{g}(\varphi(\tau))\mathbf{u}]\mathrm{d}\tau$$

Sampled-Data Control

Caltech



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Discrete State Transitions

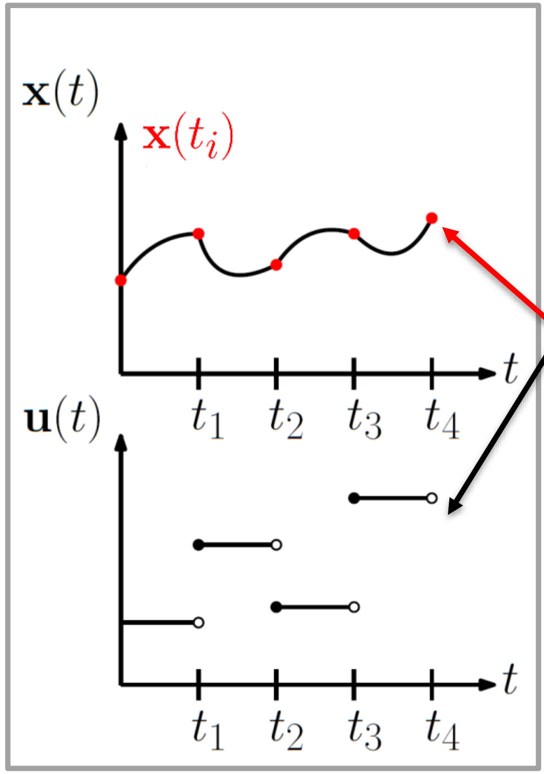
$$\mathbf{F}_T^e(\mathbf{x}, \mathbf{u}) = \mathbf{x} + \int_0^T [\mathbf{f}(\boldsymbol{\varphi}(\tau)) + \mathbf{g}(\boldsymbol{\varphi}(\tau))\mathbf{u}]\mathrm{d}\tau$$

Discrete Dynamics

$$\mathbf{x}_{i+1} = \mathbf{F}_T^e(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \qquad i \in \mathbb{Z}_{\geq 0} \qquad \mathbf{x}_0 \in \mathbb{R}^n$$

Sampled-Data Control

Discrete Time Design[8]



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$

$$t_{i+1} - t_i = T$$

Discrete State Transitions

$$\mathbf{F}^e_T(\mathbf{x}, \mathbf{u}) = \mathbf{x} + \int_0^T [\mathbf{f}(\varphi(\tau)) + \mathbf{g}(\varphi(\tau))\mathbf{u}]d\tau$$

Discrete Dynamics

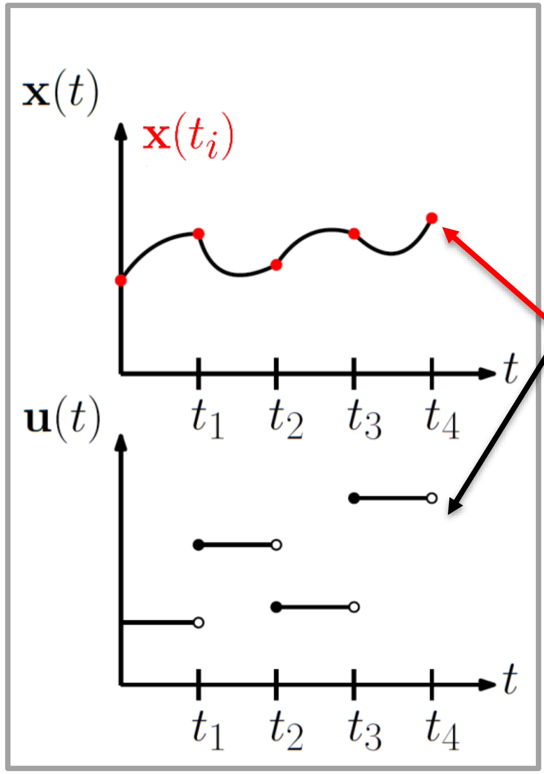
$$\mathbf{x}_{i+1} = \mathbf{F}^e_T(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \quad i \in \mathbb{Z}_{\geq 0} \quad \mathbf{x}_0 \in \mathbb{R}^n$$

[8] S. Monaco, D. Normand-Cyrot, "Advanced Tools for Nonlinear Sampled-Data Systems' Analysis and Control", 2007.

Sampled-Data Control

Discrete Time Design[8]

Caltech



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Discrete State Transitions

$$\mathbf{F}_T^e(\mathbf{x}, \mathbf{u}) = \mathbf{x} + \int_0^T [\mathbf{f}(\boldsymbol{\varphi}(\tau)) + \mathbf{g}(\boldsymbol{\varphi}(\tau))\mathbf{u}]\mathrm{d}\tau$$

Discrete Dynamics

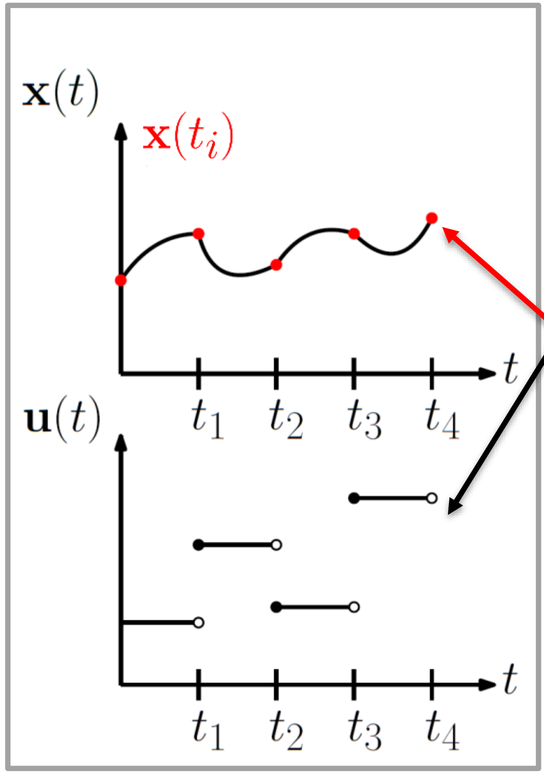
$$\mathbf{x}_{i+1} = \mathbf{F}_T^e(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \quad i \in \mathbb{Z}_{\geq 0} \quad \mathbf{x}_0 \in \mathbb{R}^n$$

What kind of guarantees?

[8] S. Monaco, D. Normand-Cyrot, "Advanced Tools for Nonlinear Sampled-Data Systems' Analysis and Control", 2007.

Sampled-Data Control

Discrete Time Design[8]



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \text{ for all } t \in [t_i, t_{i+1})$$
$$t_{i+1} - t_i = T$$

Discrete State Transitions

$$\mathbf{F}^e_T(\mathbf{x}, \mathbf{u}) = \mathbf{x} + \int_0^T [\mathbf{f}(\boldsymbol{\varphi}(\tau)) + \mathbf{g}(\boldsymbol{\varphi}(\tau))\mathbf{u}]d\tau$$

Discrete Dynamics

$$\mathbf{x}_{i+1} = \mathbf{F}^e_T(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \qquad i \in \mathbb{Z}_{\geq 0} \qquad \mathbf{x}_0 \in \mathbb{R}^n$$

What kind of guarantees?

How do we work with?

[8] S. Monaco, D. Normand-Cyrot, "Advanced Tools for Nonlinear Sampled-Data Systems' Analysis and Control", 2007.

Approximate Discrete Models

Caltech

Approximate Discrete Model

$$\mathbf{F}^{a}_{T}(\mathbf{x},\mathbf{u}) \approx \mathbf{F}^{e}_{T}(\mathbf{x},\mathbf{u})$$

Approximate Discrete Models

Caltech

Approximate Discrete Model

$$\mathbf{F}_T^a(\mathbf{x},\mathbf{u}) \approx \mathbf{F}_T^e(\mathbf{x},\mathbf{u})$$

Stability Analysis

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.
[10] D. Nešić, A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", 2004.

Approximate Discrete Models

Caltech

Approximate Discrete Model

$$\mathbf{F}_T^a(\mathbf{x},\mathbf{u}) \approx \mathbf{F}_T^e(\mathbf{x},\mathbf{u})$$

Stability Analysis

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.

[10] D. Nešić, A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", 2004.

One-Step Consistency

$$\|\mathbf{F}_T^a(\mathbf{x},\mathbf{u}) - \mathbf{F}_T^e(\mathbf{x},\mathbf{u})\| \leq T\rho(T)$$

$$T^* \in \mathbb{R}_{>0} \qquad T \in (0, T^*) \qquad \rho \in \mathcal{K}$$

Approximate Discrete Models

Caltech

Approximate Discrete Model

$$\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) \approx \mathbf{F}^e_T(\mathbf{x},\mathbf{u})$$

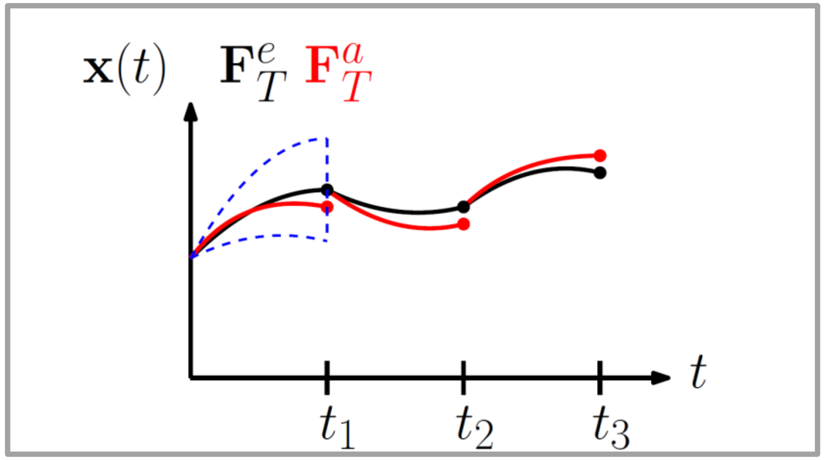
Stability Analysis

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.
[10] D. Nešić, A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", 2004.

One-Step Consistency

$$\|\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) - \mathbf{F}^e_T(\mathbf{x},\mathbf{u})\| \leq T\rho(T)$$

$T^* \in \mathbb{R}_{>0}$ $\quad T \in (0, T^*)$ $\quad \rho \in \mathcal{K}$



Approximate Discrete Models

Caltech

Approximate Discrete Model

$$\mathbf{F}_T^a(\mathbf{x},\mathbf{u}) \approx \mathbf{F}_T^e(\mathbf{x},\mathbf{u})$$

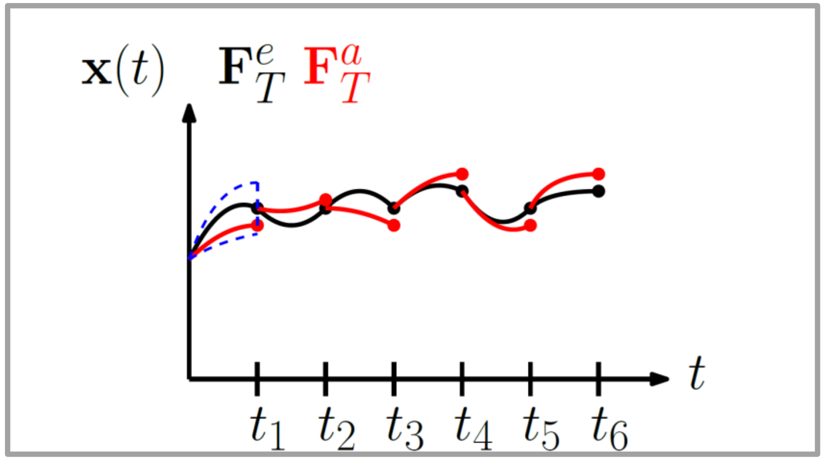
Stability Analysis

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.
[10] D. Nešić, A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", 2004.

One-Step Consistency

$$\|\mathbf{F}_T^a(\mathbf{x},\mathbf{u}) - \mathbf{F}_T^e(\mathbf{x},\mathbf{u})\| \leq T\rho(T)$$

$T^* \in \mathbb{R}_{>0}$ $\qquad T \in (0, T^*)$ $\qquad \rho \in \mathcal{K}$



Approximate Discrete Models

Caltech

Approximate Discrete Model

$$\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) \approx \mathbf{F}^e_T(\mathbf{x},\mathbf{u})$$

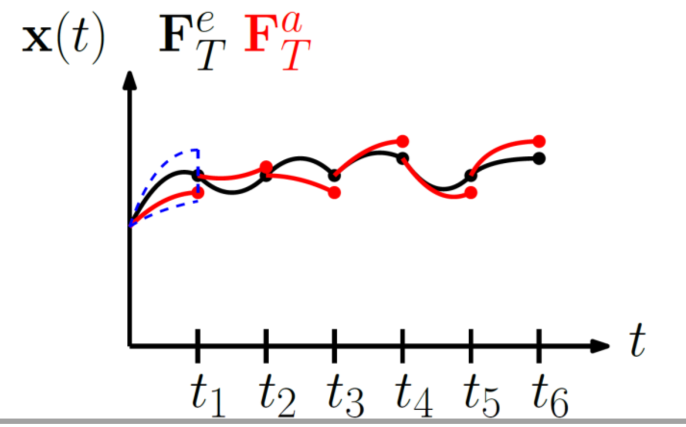
Stability Analysis

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.
[10] D. Nešić, A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", 2004.

One-Step Consistency

$$\|\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) - \mathbf{F}^e_T(\mathbf{x},\mathbf{u})\| \leq T\rho(T)$$

$$T^* \in \mathbb{R}_{>0} \qquad T \in (0, T^*) \qquad \rho \in \mathcal{K}$$



Euler Approximate Model

$$\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) = \mathbf{x} + T(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u})$$

Approximate Discrete Models

Caltech

Approximate Discrete Model

$$\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) \approx \mathbf{F}^e_T(\mathbf{x},\mathbf{u})$$

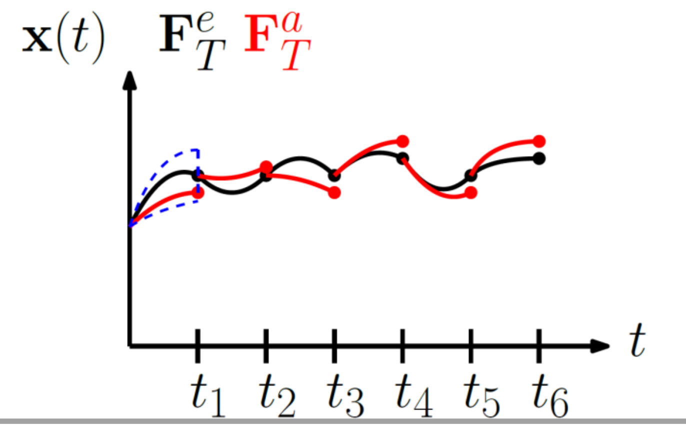
Stability Analysis

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.
[10] D. Nešić, A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", 2004.

One-Step Consistency

$$\|\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) - \mathbf{F}^e_T(\mathbf{x},\mathbf{u})\| \leq T\rho(T)$$

$T^* \in \mathbb{R}_{>0}$ $\quad T \in (0, T^*)$ $\quad \rho \in \mathcal{K}$



Euler Approximate Model

$$\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) = \mathbf{x} + T(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u})$$

$\mathbf{f},\ \mathbf{g}$ locally Lipschitz $\implies$ One-Step Consistency

Approximate Discrete Models

Caltech

Approximate Discrete Model

$$\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) \approx \mathbf{F}^e_T(\mathbf{x},\mathbf{u})$$

Stability Analysis

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.
[10] D. Nešić, A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", 2004.

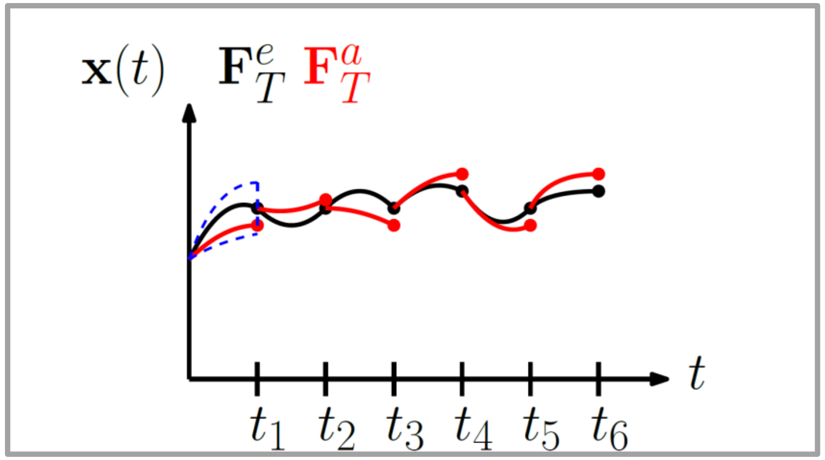
One-Step Consistency

$$\|\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) - \mathbf{F}^e_T(\mathbf{x},\mathbf{u})\| \leq T\rho(T)$$

$T^* \in \mathbb{R}_{>0}$ $\quad T \in (0, T^*)$ $\quad \rho \in \mathcal{K}$

Optimization-Based Controllers

[11] **A. Taylor**, V. Dorobantu, Y. Yue, P. Tabuada, A. Ames, "Sampled-Data Stabilization with Control Lyapunov Functions via Quadratically Constrained Quadratic Programs", 2022.



Euler Approximate Model

$$\mathbf{F}^a_T(\mathbf{x},\mathbf{u}) = \mathbf{x} + T(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u})$$

$\mathbf{f},\ \mathbf{g}$ locally Lipschitz $\implies$ One-Step Consistency

Practical Stability

Caltech

Practical Stability[9]

$$\mathbf{x}_{i+1} = \mathbf{F}_T(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \qquad i \in \mathbb{Z}_{\geq 0}$$

For each $R \in \mathbb{R}_{>0}$, there exists $T^* \in \mathbb{R}_{>0}$ s.t.

$$T \in (0, T^*) \implies \|\mathbf{x}_i\| \leq \beta(\|\mathbf{x}_0\|, iT) + R$$

$$\beta \in \mathcal{KL}$$

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.

Practical Stability

Caltech

Practical Stability[9]

$$\mathbf{x}_{i+1} = \mathbf{F}_T(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \qquad i \in \mathbb{Z}_{\geq 0}$$

For each $R \in \mathbb{R}_{>0}$, there exists $T^* \in \mathbb{R}_{>0}$ s.t.

$$T \in (0, T^*) \implies \|\mathbf{x}_i\| \leq \beta(\|\mathbf{x}_0\|, iT) + R$$

$$\beta \in \mathcal{KL}$$

Only at sample times

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.

Practical Stability

Caltech

Practical Stability[9]

$$\mathbf{x}_{i+1} = \mathbf{F}_T(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \qquad i \in \mathbb{Z}_{\geq 0}$$

For each $R \in \mathbb{R}_{>0}$, there exists $T^* \in \mathbb{R}_{>0}$ s.t.

$$T \in (0, T^*) \implies \|\mathbf{x}_i\| \leq \beta(\|\mathbf{x}_0\|, iT) + R$$

$$\beta \in \mathcal{KL}$$

Only at sample times

Relaxation decreases with sample period

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.

Practical Stability

Caltech

Practical Stability[9]

$$\mathbf{x}_{i+1} = \mathbf{F}_T(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \qquad i \in \mathbb{Z}_{\geq 0}$$

For each $R \in \mathbb{R}_{>0}$, there exists $T^* \in \mathbb{R}_{>0}$ s.t.

$$T \in (0, T^*) \implies \|\mathbf{x}_i\| \leq \beta(\|\mathbf{x}_0\|, iT) + R$$

$$\beta \in \mathcal{KL}$$

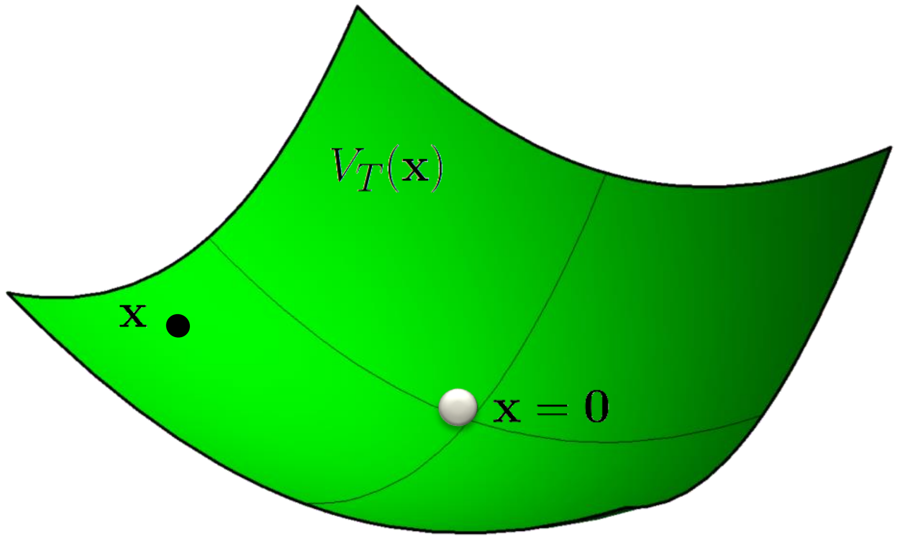
Equi-Lipschitz Lyapunov Function[9]

$$\alpha_1(\|\mathbf{x}\|) \leq V_T(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$$

$$V_T(\mathbf{F}_T(\mathbf{x}, \mathbf{k}_T(\mathbf{x}))) - V_T(\mathbf{x}) \leq -T\alpha_3(\|\mathbf{x}\|)$$

$$|V_T(\mathbf{x}) - V_T(\mathbf{y})| \leq M\|\mathbf{x} - \mathbf{y}\|$$

$$T^* \in \mathbb{R}_{>0} \qquad T \in (0, T^*) \qquad \alpha_i \in \mathcal{K}$$



V_T for $\mathbf{F}_T \implies \mathbf{F}_T$ stable

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.

Practical Stability

Caltech

Practical Stability[9]

$$\mathbf{x}_{i+1} = \mathbf{F}_T(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \qquad i \in \mathbb{Z}_{\geq 0}$$

For each $R \in \mathbb{R}_{>0}$, there exists $T^* \in \mathbb{R}_{>0}$ s.t.

$$T \in (0, T^*) \implies \|\mathbf{x}_i\| \leq \beta(\|\mathbf{x}_0\|, iT) + R$$

$$\beta \in \mathcal{KL}$$

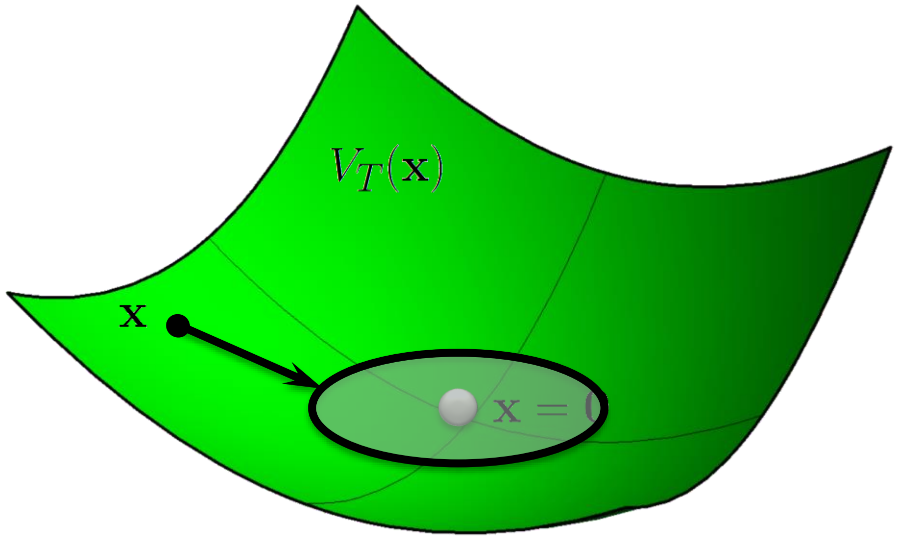
Equi-Lipschitz Lyapunov Function[9]

$$\alpha_1(\|\mathbf{x}\|) \leq V_T(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$$

$$V_T(\mathbf{F}_T(\mathbf{x}, \mathbf{k}_T(\mathbf{x}))) - V_T(\mathbf{x}) \leq -T\alpha_3(\|\mathbf{x}\|)$$

$$|V_T(\mathbf{x}) - V_T(\mathbf{y})| \leq M\|\mathbf{x} - \mathbf{y}\|$$

$$T^* \in \mathbb{R}_{>0} \qquad T \in (0, T^*) \qquad \alpha_i \in \mathcal{K}$$



$$V_T \text{ for } \mathbf{F}_T \implies \mathbf{F}_T \text{ stable}$$

V_T for $\mathbf{F}_T^a$ + One-Step Consistency $\implies \mathbf{F}_T^e$ practically stable

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.

Practical Stability

Design with approximation endows exact system with guarantees!

Caltech

Practical Stability[9]

$$\mathbf{x}_{i+1} = \mathbf{F}_T(\mathbf{x}_i, \mathbf{k}_T(\mathbf{x}_i)) \qquad i \in \mathbb{Z}_{\geq 0}$$

For each $R \in \mathbb{R}_{>0}$, there exists $T^* \in \mathbb{R}_{>0}$ s.t.

$$T \in (0, T^*) \implies \|\mathbf{x}_i\| \leq \beta(\|\mathbf{x}_0\|, iT) + R$$

$$\beta \in \mathcal{KL}$$

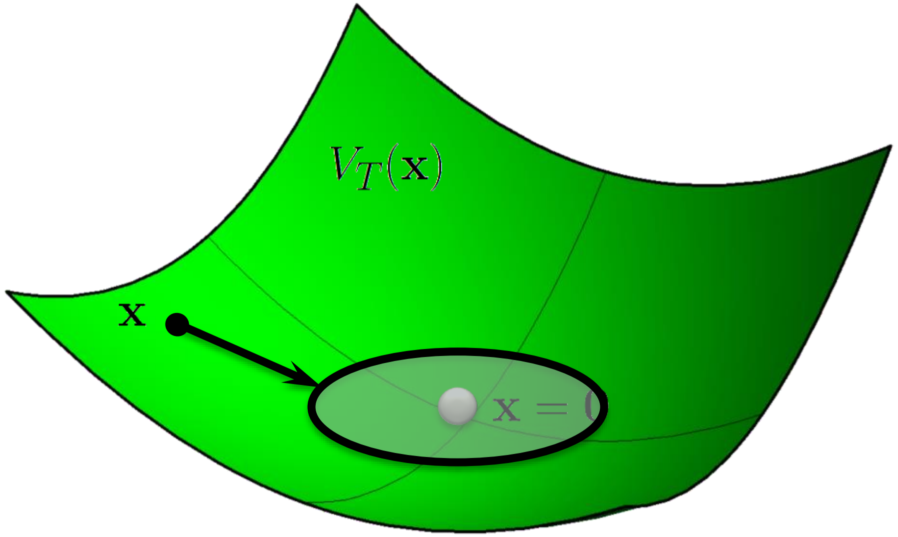
Equi-Lipschitz Lyapunov Function[9]

$$\alpha_1(\|\mathbf{x}\|) \leq V_T(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$$

$$V_T(\mathbf{F}_T(\mathbf{x}, \mathbf{k}_T(\mathbf{x}))) - V_T(\mathbf{x}) \leq -T\alpha_3(\|\mathbf{x}\|)$$

$$|V_T(\mathbf{x}) - V_T(\mathbf{y})| \leq M\|\mathbf{x} - \mathbf{y}\|$$

$$T^* \in \mathbb{R}_{>0} \qquad T \in (0, T^*) \qquad \alpha_i \in \mathcal{K}$$



$$V_T \text{ for } \mathbf{F}_T \implies \mathbf{F}_T \text{ stable}$$

$$\begin{array}{c} V_T \text{ for } \mathbf{F}_T^a \\ + \\ \text{One-Step Consistency} \end{array} \implies \mathbf{F}_T^e \text{ practically stable}$$

[9] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999.

Relevant Questions

Caltech

1. What is an appropriate definition of **practical safety**?

2. How will practical safety connect with the notion of a **sampled-data** Barrier Function?

3. What sort of approximate discrete transition maps should we use with sampled-data Control Barrier Functions?

Practical Safety

Caltech

Sampled-Data Forward Invariance

$\mathcal{C}$ forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$ if

$$\mathbf{x}_0 \in \mathcal{C} \implies \mathbf{x}_i \in \mathcal{C} \quad i \in \mathbb{Z}_{\geq 0}$$

Practical Safety

Caltech

Only at sample times

Sampled-Data Forward Invariance

$\mathcal{C}$ forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$ if

$$\mathbf{x}_0 \in \mathcal{C} \implies \mathbf{x}_i \in \mathcal{C} \quad i \in \mathbb{Z}_{\geq 0}$$

Practical Safety

Caltech

Only at sample times

Sampled-Data Forward Invariance

$\mathcal{C}$ forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$ if

$$\mathbf{x}_0 \in \mathcal{C} \implies \mathbf{x}_i \in \mathcal{C} \quad i \in \mathbb{Z}_{\geq 0}$$

Practical Safety

For each $R \in \mathbb{R}_{>0}$ there exists $T^* \in \mathbb{R}_{>0}$ s.t. for each $T \in (0, T^*)$ there is a set $\mathcal{C}_T \subset \mathbb{R}^n$ with $\mathcal{C} \subseteq \mathcal{C}_T \subseteq \mathcal{C} \oplus \overline{B}_R$ that is forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$

Practical Safety

Caltech

Only at sample times

Sampled-Data Forward Invariance

$\mathcal{C}$ forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$ if
$$\mathbf{x}_0 \in \mathcal{C} \implies \mathbf{x}_i \in \mathcal{C} \quad i \in \mathbb{Z}_{\geq 0}$$

Practical Safety

For each $R \in \mathbb{R}_{>0}$ there exists $T^* \in \mathbb{R}_{>0}$ s.t. for each $T \in (0, T^*)$ there is a set $\mathcal{C}_T \subset \mathbb{R}^n$ with $\mathcal{C} \subseteq \mathcal{C}_T \subseteq \mathcal{C} \oplus \overline{B}_R$ that is forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$

Relaxation decreases with sample period

Practical Safety

Caltech

Sampled-Data Forward Invariance

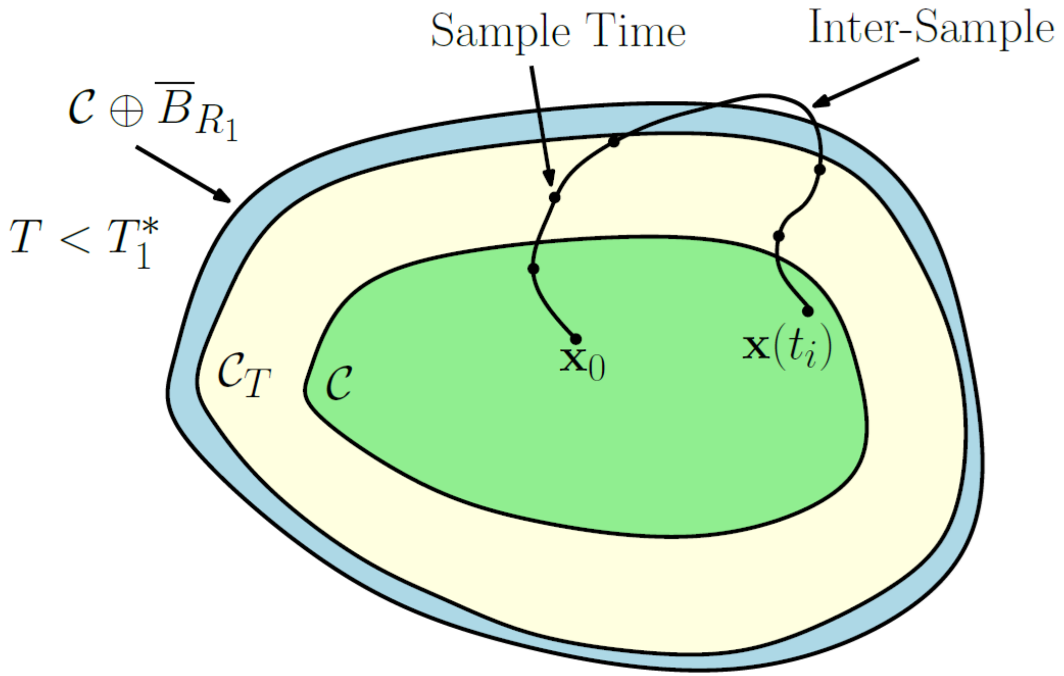
$\mathcal{C}$ forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$ if
$\mathbf{x}_0 \in \mathcal{C} \implies \mathbf{x}_i \in \mathcal{C} \quad i \in \mathbb{Z}_{\geq 0}$

Only at sample times

Practical Safety

For each $R \in \mathbb{R}_{>0}$ there exists $T^* \in \mathbb{R}_{>0}$ s.t. for each $T \in (0, T^*)$ there is a set $\mathcal{C}_T \subset \mathbb{R}^n$ with $\mathcal{C} \subseteq \mathcal{C}_T \subseteq \mathcal{C} \oplus \overline{B}_R$ that is forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$

Relaxation decreases with sample period



Practical Safety

Caltech

Only at sample times

Sampled-Data Forward Invariance

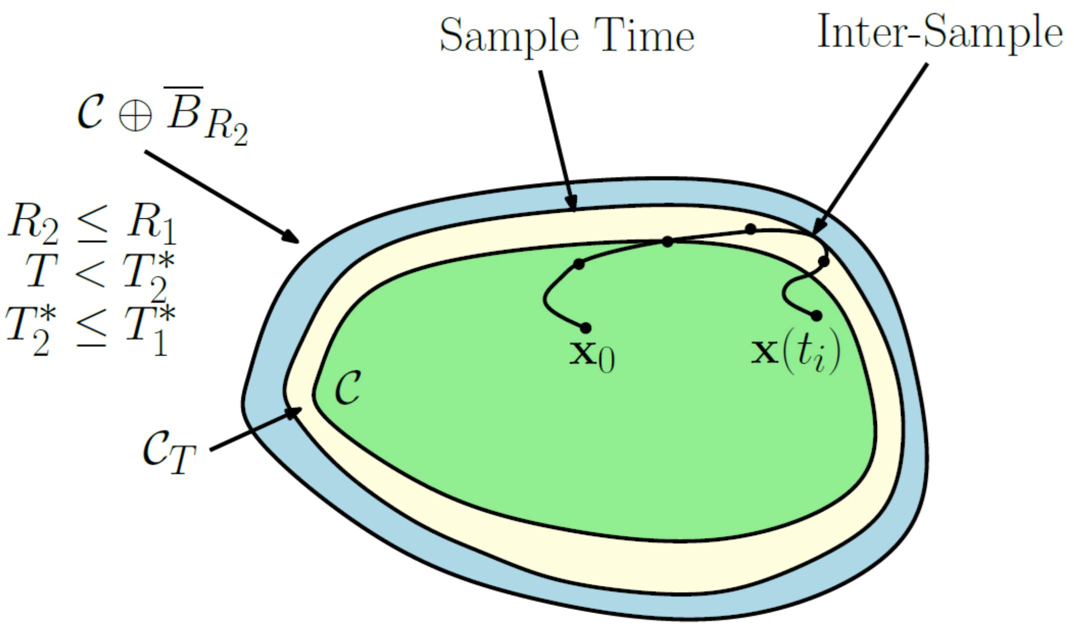
$\mathcal{C}$ forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$ if

$$\mathbf{x}_0 \in \mathcal{C} \implies \mathbf{x}_i \in \mathcal{C} \quad i \in \mathbb{Z}_{\geq 0}$$

Practical Safety

For each $R \in \mathbb{R}_{>0}$ there exists $T^* \in \mathbb{R}_{>0}$ s.t. for each $T \in (0, T^*)$ there is a set $\mathcal{C}_T \subset \mathbb{R}^n$ with $\mathcal{C} \subseteq \mathcal{C}_T \subseteq \mathcal{C} \oplus \overline{B}_R$ that is forward invariant for $(\mathbf{k}_T, \mathbf{F}_T)$

Relaxation decreases with sample period



Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

$$T^* \in \mathbb{R}_{>0} \quad \alpha \in \mathcal{K}^{\mathrm{e}}_{\infty} \quad \epsilon, M \in \mathbb{R}_{>0}$$

Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

$$T^* \in \mathbb{R}_{>0} \quad \alpha \in \mathcal{K}^{\mathrm{e}}_{\infty} \quad \epsilon, M \in \mathbb{R}_{>0}$$

For all $T \in (0, T^*)$

1) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h_T(\mathbf{x}) \geq 0\}$

0-Superlevel Safe Set

Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

$$T^* \in \mathbb{R}_{>0} \quad \alpha \in \mathcal{K}^{\mathrm{e}}_{\infty} \quad \epsilon, M \in \mathbb{R}_{>0}$$

For all $T \in (0, T^*)$

1) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h_T(\mathbf{x}) \geq 0\}$

2) $T\alpha(h_T(\mathbf{x})) \leq h_T(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C}$

0-Superlevel Safe Set

Discrete Barrier Requirement

Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

$$T^* \in \mathbb{R}_{>0} \quad \alpha \in \mathcal{K}^{\mathrm{e}}_{\infty} \quad \epsilon, M \in \mathbb{R}_{>0}$$

For all $T \in (0, T^*)$

1) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h_T(\mathbf{x}) \geq 0\}$

2) $T\alpha(h_T(\mathbf{x})) \leq h_T(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C}$

3) $|h_T(\mathbf{y}) - h_T(\mathbf{x})| \leq M\|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{C} \oplus \overline{B}_\epsilon$

0-Superlevel Safe Set

Discrete Barrier Requirement

Lipschitz Continuity

Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

$T^* \in \mathbb{R}_{>0} \quad \alpha \in \mathcal{K}_\infty^{\mathrm{e}} \quad \epsilon, M \in \mathbb{R}_{>0}$

For all $T \in (0, T^*)$

1) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h_T(\mathbf{x}) \geq 0\}$

2) $T\alpha(h_T(\mathbf{x})) \leq h_T(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C}$

3) $|h_T(\mathbf{y}) - h_T(\mathbf{x})| \leq M\|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{C} \oplus \overline{B}_\epsilon$

0-Superlevel Safe Set

Discrete Barrier Requirement

Lipschitz Continuity

+

Coercivity

$$d_{\mathcal{C}}(\mathbf{x}) = \inf_{\mathbf{y} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|$$

4) For each $\eta \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ s.t.

$$d_{\mathcal{C}}(\mathbf{x}) > \eta \implies h_T(\mathbf{x}) < -\delta$$

for all $\mathbf{x} \in \mathcal{C} \oplus \overline{B}_\epsilon$

Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

$$T^* \in \mathbb{R}_{>0} \quad \alpha \in \mathcal{K}_\infty^{\mathrm{e}} \quad \epsilon, M \in \mathbb{R}_{>0}$$

For all $T \in (0, T^*)$

1) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h_T(\mathbf{x}) \geq 0\}$

2) $T\alpha(h_T(\mathbf{x})) \leq h_T(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C}$

3) $|h_T(\mathbf{y}) - h_T(\mathbf{x})| \leq M\|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{C} \oplus \overline{B}_\epsilon$

0-Superlevel Safe Set

Discrete Barrier Requirement

Lipschitz Continuity

+

Coercivity

$$d_{\mathcal{C}}(\mathbf{x}) = \inf_{\mathbf{y} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|$$

4) For each $\eta \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ s.t.

$$d_{\mathcal{C}}(\mathbf{x}) > \eta \implies h_T(\mathbf{x}) < -\delta$$

for all $\mathbf{x} \in \mathcal{C} \oplus \overline{B}_\epsilon$

0 is regular value of h_T

Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

$$T^* \in \mathbb{R}_{>0} \quad \alpha \in \mathcal{K}_\infty^{\mathrm{e}} \quad \epsilon, M \in \mathbb{R}_{>0}$$

For all $T \in (0, T^*)$

1) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h_T(\mathbf{x}) \geq 0\}$

2) $T\alpha(h_T(\mathbf{x})) \leq h_T(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C}$

3) $|h_T(\mathbf{y}) - h_T(\mathbf{x})| \leq M\|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{C} \oplus \overline{B}_\epsilon$

0-Superlevel Safe Set

Discrete Barrier Requirement

Lipschitz Continuity

+

Coercivity

$$d_{\mathcal{C}}(\mathbf{x}) = \inf_{\mathbf{y} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|$$

4) For each $\eta \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ s.t.

$$d_{\mathcal{C}}(\mathbf{x}) > \eta \implies h_T(\mathbf{x}) < -\delta$$

for all $\mathbf{x} \in \mathcal{C} \oplus \overline{B}_\epsilon$

0 is regular value of h_T

Sampled-Data Control Barrier Function

$$\sup_{\mathbf{u} \in \mathbb{R}^m} h_T(\mathbf{F}_T(\mathbf{x}, \mathbf{u})) - h_T(\mathbf{x}) > -T\alpha(h_T(\mathbf{x}))$$

Sampled-Data Barrier Functions

Caltech

Sampled-Data Barrier Function Candidate

$$\{h_T \mid T \in \mathbb{R}_{>0}\}$$

$$T^* \in \mathbb{R}_{>0} \quad \alpha \in \mathcal{K}_\infty^{\mathrm{e}} \quad \epsilon, M \in \mathbb{R}_{>0}$$

For all $T \in (0, T^*)$

1) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h_T(\mathbf{x}) \geq 0\}$

2) $T\alpha(h_T(\mathbf{x})) \leq h_T(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C}$

3) $|h_T(\mathbf{y}) - h_T(\mathbf{x})| \leq M\|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{C} \oplus \overline{B}_\epsilon$

0-Superlevel Safe Set

Discrete Barrier Requirement

Lipschitz Continuity

+

Coercivity

$$d_{\mathcal{C}}(\mathbf{x}) = \inf_{\mathbf{y} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|$$

4) For each $\eta \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ s.t.

$$d_{\mathcal{C}}(\mathbf{x}) > \eta \implies h_T(\mathbf{x}) < -\delta$$

for all $\mathbf{x} \in \mathcal{C} \oplus \overline{B}_\epsilon$

0 is regular value of h_T

Sampled-Data Control Barrier Function

$$\sup_{\mathbf{u} \in \mathbb{R}^m} h_T(\mathbf{F}_T(\mathbf{x}, \mathbf{u})) - h_T(\mathbf{x}) > -T\alpha(h_T(\mathbf{x}))$$

Sampled-Data Barrier Function

$$h_T(\mathbf{F}_T(\mathbf{x}, \mathbf{k}_T(\mathbf{x}))) - h_T(\mathbf{x}) \geq -T\alpha(h_T(\mathbf{x}))$$

Sampled-Data Barrier Functions

Caltech

Practical Safety Result

Theorem 16. *Consider a set* $\mathcal{C} \subseteq \mathbb{R}^n$ *and a family of controllers* $\{\mathbf{k}_T \mid T \in I\}$*. Suppose that:*

1) *There exists a family of Sampled-Data Barrier Functions on* $\mathcal{C}$ *for a family* $\{(\mathbf{k}_T, \mathbf{F}_T) \mid T \in I\}$.
2) *There exists an* $\varepsilon' \in \mathbb{R}_{>0}$ *such that the family* $\{(\mathbf{k}_T, \mathbf{F}_T) \mid T \in I\}$ *is one-step consistent with the exact family* $\{(\mathbf{k}_T, \mathbf{F}^e_T) \mid T \in I\}$ *over the set* $\mathcal{C} \oplus \overline{B}_{\varepsilon'}$.

Then the exact family $\{(\mathbf{k}_T, \mathbf{F}^e_T) \mid T \in I\}$ *is practically safe with respect to* $\mathcal{C}$.

Sampled-Data Barrier Functions

Caltech

Practical Safety Result

Theorem 16. *Consider a set* $\mathcal{C} \subseteq \mathbb{R}^n$ *and a family of controllers* $\{\mathbf{k}_T \mid T \in I\}$. *Suppose that:*

1) *There exists a family of Sampled-Data Barrier Functions on* $\mathcal{C}$ *for a family* $\{(\mathbf{k}_T, \mathbf{F}_T) \mid T \in I\}$.
2) *There exists an* $\varepsilon' \in \mathbb{R}_{>0}$ *such that the family* $\{(\mathbf{k}_T, \mathbf{F}_T) \mid T \in I\}$ *is one-step consistent with the exact family* $\{(\mathbf{k}_T, \mathbf{F}_T^e) \mid T \in I\}$ *over the set* $\mathcal{C} \oplus \overline{B}_{\varepsilon'}$.

Then the exact family $\{(\mathbf{k}_T, \mathbf{F}_T^e) \mid T \in I\}$ *is practically safe with respect to* $\mathcal{C}$.

Designs using discrete approximation provide theoretical safety guarantees

Towards Sampled-Data CBFs

Caltech

Double Integrator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

Towards Sampled-Data CBFs

Caltech

Double Integrator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

Sampled-Data Barrier Function Candidate

$$h_T(\mathbf{x}) = x_1$$

Towards Sampled-Data CBFs

Caltech

Double Integrator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

Euler Approximation

$$\mathbf{F}_T^{a,1}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} x_1 + T x_2 \\ x_2 + T u \end{bmatrix}$$

Sampled-Data Barrier Function Candidate

$$h_T(\mathbf{x}) = x_1$$

Towards Sampled-Data CBFs

Caltech

Double Integrator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

Sampled-Data Barrier Function Candidate

$$h_T(\mathbf{x}) = x_1$$

Euler Approximation

$$\mathbf{F}_T^{a,1}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} x_1 + Tx_2 \\ x_2 + Tu \end{bmatrix}$$

SD-CBF Evolution

$$h_T\left(\mathbf{F}_T^{a,1}(\mathbf{x}, \mathbf{u})\right) = x_1 + Tx_2$$

Towards Sampled-Data CBFs

Caltech

Double Integrator

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2 \\ u\end{bmatrix}$$

Sampled-Data Barrier Function Candidate

$$h_T(\mathbf{x}) = x_1$$

Euler Approximation

$$\mathbf{F}_T^{a,1}(\mathbf{x},\mathbf{u}) = \begin{bmatrix}x_1 + Tx_2 \\ x_2 + Tu\end{bmatrix}$$

SD-CBF Evolution

$$h_T\left(\mathbf{F}_T^{a,1}(\mathbf{x},\mathbf{u})\right) = x_1 + Tx_2$$

Input does not appear!

Towards Sampled-Data CBFs

Caltech

Double Integrator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

Euler Approximation

$$\mathbf{F}_T^{a,1}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} x_1 + Tx_2 \\ x_2 + Tu \end{bmatrix}$$

Sampled-Data Barrier Function Candidate

$$h_T(\mathbf{x}) = x_1$$

SD-CBF Evolution

$$h_T\left(\mathbf{F}_T^{a,1}(\mathbf{x},\mathbf{u})\right) = x_1 + Tx_2$$

Input does not appear!

2nd Order Approximation

$$\mathbf{F}_T^{a,2}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} x_1 + Tx_2 + \frac{1}{2}T^2u \\ x_2 + Tu \end{bmatrix}$$

Towards Sampled-Data CBFs

Caltech

Double Integrator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

Euler Approximation

$$\mathbf{F}_T^{a,1}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} x_1 + Tx_2 \\ x_2 + Tu \end{bmatrix}$$

Sampled-Data Barrier Function Candidate

$$h_T(\mathbf{x}) = x_1$$

SD-CBF Evolution

$$h_T\left(\mathbf{F}_T^{a,1}(\mathbf{x},\mathbf{u})\right) = x_1 + Tx_2$$

Input does not appear!

2nd Order Approximation

$$\mathbf{F}_T^{a,2}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} x_1 + Tx_2 + \frac{1}{2}T^2u \\ x_2 + Tu \end{bmatrix}$$

SD-CBF Evolution

$$h_T\left(\mathbf{F}_T^{a,2}(\mathbf{x},\mathbf{u})\right) = x_1 + Tx_2 + \frac{1}{2}T^2u$$

Towards Sampled-Data CBFs

Caltech

Double Integrator

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2 \\ u\end{bmatrix}$$

Euler Approximation

$$\mathbf{F}_T^{a,1}(\mathbf{x},\mathbf{u}) = \begin{bmatrix}x_1 + Tx_2 \\ x_2 + Tu\end{bmatrix}$$

Sampled-Data Barrier Function Candidate

$$h_T(\mathbf{x}) = x_1$$

SD-CBF Evolution

$$h_T\left(\mathbf{F}_T^{a,1}(\mathbf{x},\mathbf{u})\right) = x_1 + Tx_2$$

Input does not appear!

2nd Order Approximation

$$\mathbf{F}_T^{a,2}(\mathbf{x},\mathbf{u}) = \begin{bmatrix}x_1 + Tx_2 + \frac{1}{2}T^2 u \\ x_2 + Tu\end{bmatrix}$$

SD-CBF Evolution

$$h_T\left(\mathbf{F}_T^{a,2}(\mathbf{x},\mathbf{u})\right) = x_1 + Tx_2 + \frac{1}{2}T^2 u$$

Input appears!

Runge-Kutta Approximation

Caltech

Runge-Kutta Approximation

$$\mathbf{F}_T^{a,p} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \qquad p \in \mathbb{N}$$

$$\mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{u}) = \mathbf{x} + T \textstyle\sum_{i=1}^{p} b_i(\mathbf{f}(\mathbf{z}_i) + \mathbf{g}(\mathbf{z}_i)\mathbf{u})$$

$$\mathbf{z}_i = \mathbf{x} + T \textstyle\sum_{j=1}^{i-1} a_{i,j}(\mathbf{f}(\mathbf{z}_j) + \mathbf{g}(\mathbf{z}_j)\mathbf{u})$$

$$\mathbf{z}_1 = \mathbf{x}$$

$$b_1, \dots, b_p \in \mathbb{R}_{\geq 0} \qquad \textstyle\sum_{i=1}^{p} b_i = 1$$

$$a_{i,j} \in \mathbb{R}$$

Runge-Kutta Approximation

Caltech

Runge-Kutta Approximation

$$\mathbf{F}_T^{a,p} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \qquad p \in \mathbb{N}$$

$$\mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{u}) = \mathbf{x} + T \sum_{i=1}^{p} b_i(\mathbf{f}(\mathbf{z}_i) + \mathbf{g}(\mathbf{z}_i)\mathbf{u})$$

$$\mathbf{z}_i = \mathbf{x} + T \sum_{j=1}^{i-1} a_{i,j}(\mathbf{f}(\mathbf{z}_j) + \mathbf{g}(\mathbf{z}_j)\mathbf{u})$$

$$\mathbf{z}_1 = \mathbf{x}$$

$$b_1, \dots, b_p \in \mathbb{R}_{\geq 0} \qquad \sum_{i=1}^{p} b_i = 1$$

$$a_{i,j} \in \mathbb{R}$$

One-Step Consistency

$$\|\mathbf{F}_T^e(\mathbf{x}, \mathbf{k}_T(\mathbf{x})) - \mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{k}_T(\mathbf{x}))\| \leq T\rho(T)$$

$$T^* \in I \qquad T \in (0, T^*) \qquad \rho \in \mathcal{K}_\infty$$

Runge-Kutta Approximation

Caltech

Runge-Kutta Approximation

$$\mathbf{F}_T^{a,p} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \qquad p \in \mathbb{N}$$

$$\mathbf{F}_T^{a,p}(\mathbf{x},\mathbf{u}) = \mathbf{x} + T\textstyle\sum_{i=1}^p b_i(\mathbf{f}(\mathbf{z}_i) + \mathbf{g}(\mathbf{z}_i)\mathbf{u})$$

$$\mathbf{z}_i = \mathbf{x} + T\textstyle\sum_{j=1}^{i-1} a_{i,j}(\mathbf{f}(\mathbf{z}_j) + \mathbf{g}(\mathbf{z}_j)\mathbf{u})$$

$$\mathbf{z}_1 = \mathbf{x}$$

$$b_1, \dots, b_p \in \mathbb{R}_{\geq 0} \qquad \textstyle\sum_{i=1}^p b_i = 1$$

$$a_{i,j} \in \mathbb{R}$$

One-Step Consistency

$$\|\mathbf{F}_T^e(\mathbf{x}, \mathbf{k}_T(\mathbf{x})) - \mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{k}_T(\mathbf{x}))\| \leq T\rho(T)$$

$$T^* \in I \qquad T \in (0, T^*) \qquad \rho \in \mathcal{K}_\infty$$

$\Uparrow$

Sufficient Conditions

$\mathbf{f}, \mathbf{g}$ locally Lipschitz continuous $\qquad K \subset \mathbb{R}^n$ compact

$T_1 \in I \quad \mathbf{k}_T$ bounded by M_K for $T \in (0, T_1)$

Preservation of Convexity

Caltech

Structured System

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \end{bmatrix}$$

Preservation of Convexity

Caltech

Structured System

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \end{bmatrix}$$

Partially Concave Barrier Function

$$\widetilde{h}_T : \mathbb{R}^q \to \mathbb{R} \qquad q \leq n$$

$$h_T(\mathbf{x}) = \widetilde{h}_T(x_1, \ldots, x_q)$$

$\widetilde{h}_T$ concave with respect to last argument

Preservation of Convexity

Caltech

Structured System

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \end{bmatrix}$$

Partially Concave Barrier Function

$$\widetilde{h}_T : \mathbb{R}^q \to \mathbb{R} \qquad q \leq n$$

$$h_T(\mathbf{x}) = \widetilde{h}_T(x_1, \dots, x_q)$$

$\widetilde{h}_T$ concave with respect to last argument

Runge-Kutta Approximation Order

$\mathbf{F}_T^{a,p}$ Runge-Kutta approximation of order

$$p = n - q + 1$$

Preservation of Convexity

Caltech

Structured System

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \end{bmatrix}$$

Partially Concave Barrier Function

$$\widetilde{h}_T : \mathbb{R}^q \to \mathbb{R} \qquad q \leq n$$

$$h_T(\mathbf{x}) = \widetilde{h}_T(x_1, \dots, x_q)$$

$\widetilde{h}_T$ concave with respect to last argument

Runge-Kutta Approximation Order

$\mathbf{F}_T^{a,p}$ Runge-Kutta approximation of order $p = n - q + 1$

Constraint Convexity

$$\phi_T : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

$$\phi_T(\mathbf{x}, \mathbf{u}) = -h_T(\mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{u})) + h_T(\mathbf{x}) - T\alpha(h_T(\mathbf{x}))$$

ϕ_T is convex with respect to its second argument

Preservation of Convexity

Caltech

Structured System

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \end{bmatrix}$$

Partially Concave Barrier Function

$$\widetilde{h}_T : \mathbb{R}^q \to \mathbb{R} \qquad q \leq n$$

$$h_T(\mathbf{x}) = \widetilde{h}_T(x_1, \dots, x_q)$$

$\widetilde{h}_T$ concave with respect to last argument

Runge-Kutta Approximation Order

$\mathbf{F}_T^{a,p}$ Runge-Kutta approximation of order $p = n - q + 1$

Constraint Convexity

$$\phi_T : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

$$\phi_T(\mathbf{x}, \mathbf{u}) = -h_T(\mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{u})) + h_T(\mathbf{x}) - T\alpha(h_T(\mathbf{x}))$$

ϕ_T is convex with respect to its second argument

Optimization-Based Controller

$$\mathbf{k}_T(\mathbf{x}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\mathrm{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t. } h_T(\mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{u})) - h_T(\mathbf{x}) \geq -T\alpha(h_T(\mathbf{x}))$$

Preservation of Convexity

Caltech

Structured System

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \end{bmatrix}$$

Partially Concave Barrier Function

$$\widetilde{h}_T : \mathbb{R}^q \to \mathbb{R} \qquad q \leq n$$

$$h_T(\mathbf{x}) = \widetilde{h}_T(x_1, \dots, x_q)$$

$\widetilde{h}_T$ concave with respect to last argument

Runge-Kutta Approximation Order

$\mathbf{F}_T^{a,p}$ Runge-Kutta approximation of order

$$p = n - q + 1$$

Constraint Convexity

$$\phi_T : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

$$\phi_T(\mathbf{x}, \mathbf{u}) = -h_T(\mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{u})) + h_T(\mathbf{x}) - T\alpha(h_T(\mathbf{x}))$$

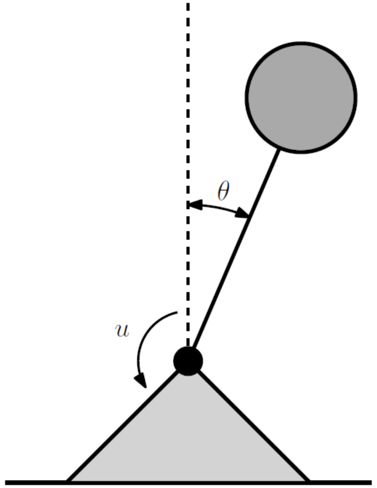
ϕ_T is convex with respect to its second argument

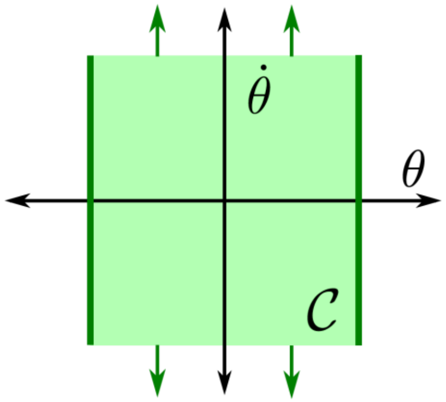
Optimization-Based Controller

$$\mathbf{k}_T(\mathbf{x}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\mathrm{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t. } h_T(\mathbf{F}_T^{a,p}(\mathbf{x}, \mathbf{u})) - h_T(\mathbf{x}) \geq -T\alpha(h_T(\mathbf{x}))$$

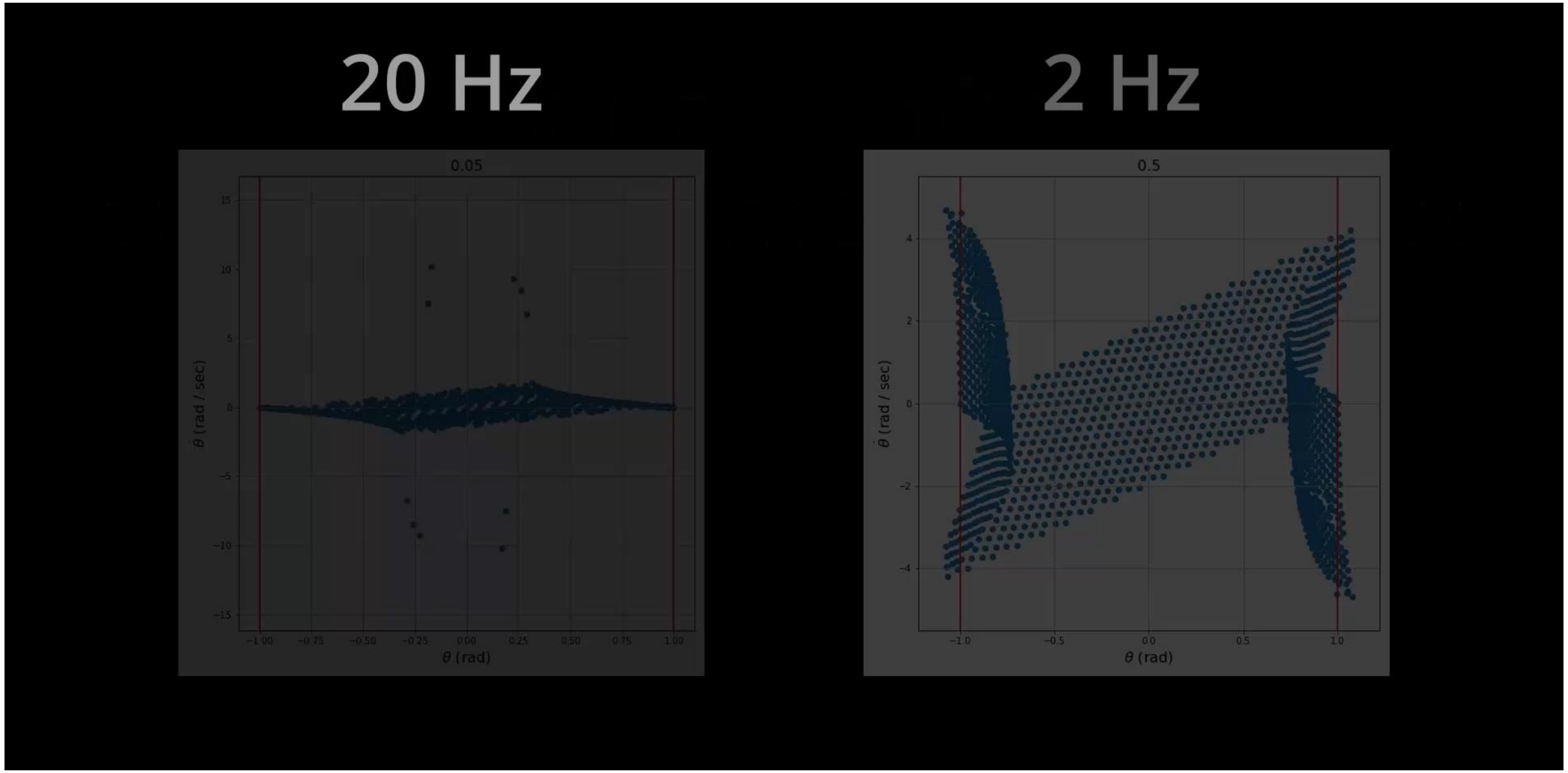
Convex optimization problem class depends on safe set geometry

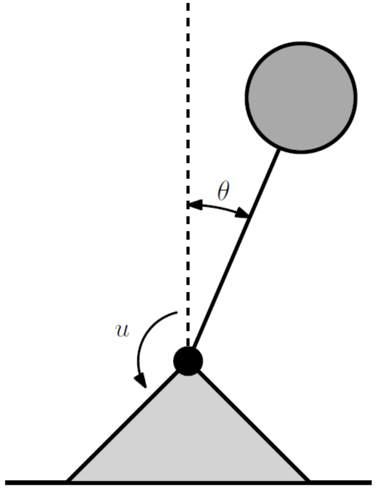
Inverted Pendulum

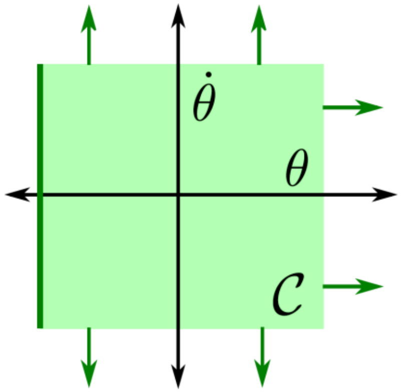




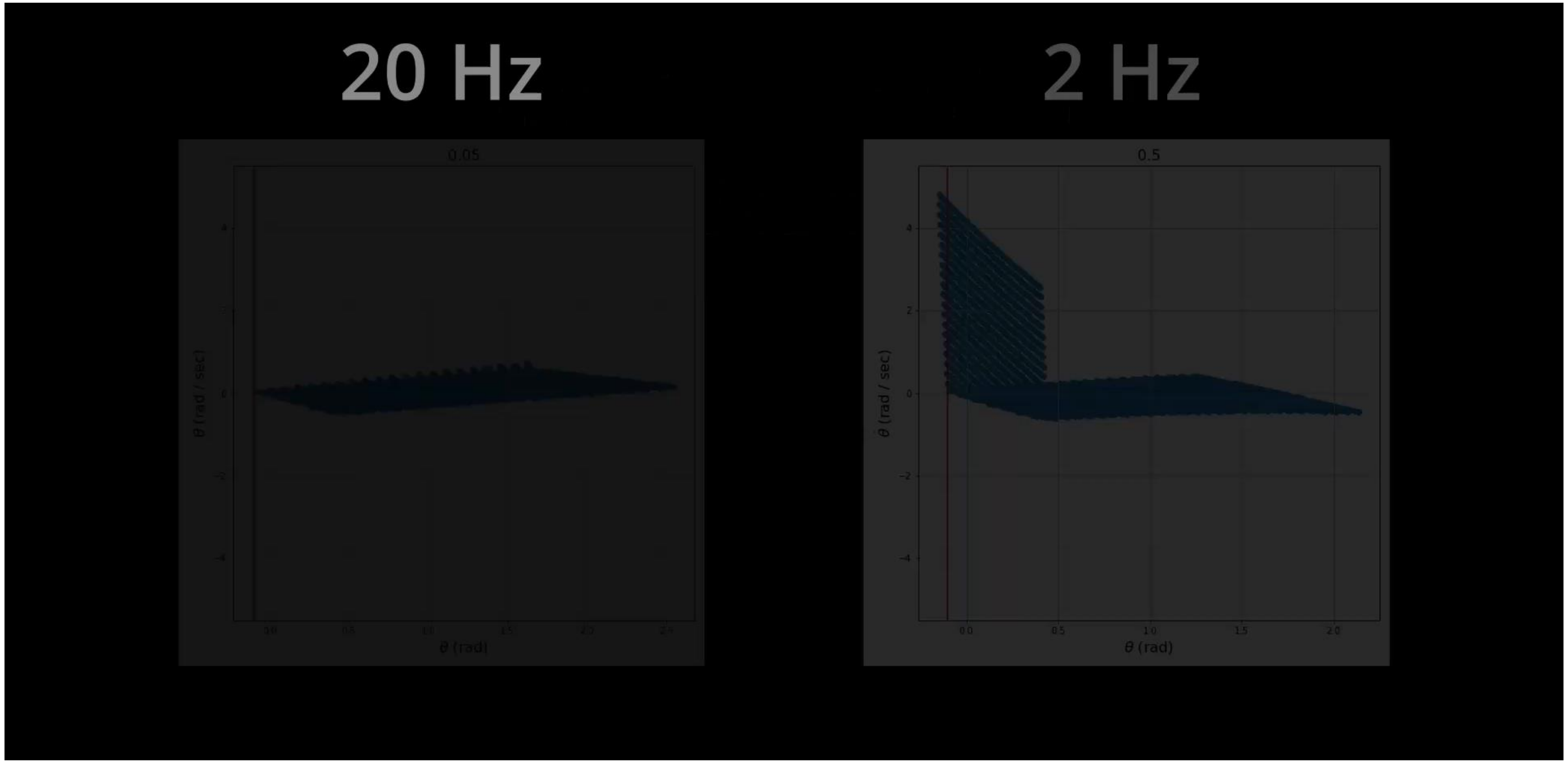
$$p = 2$$



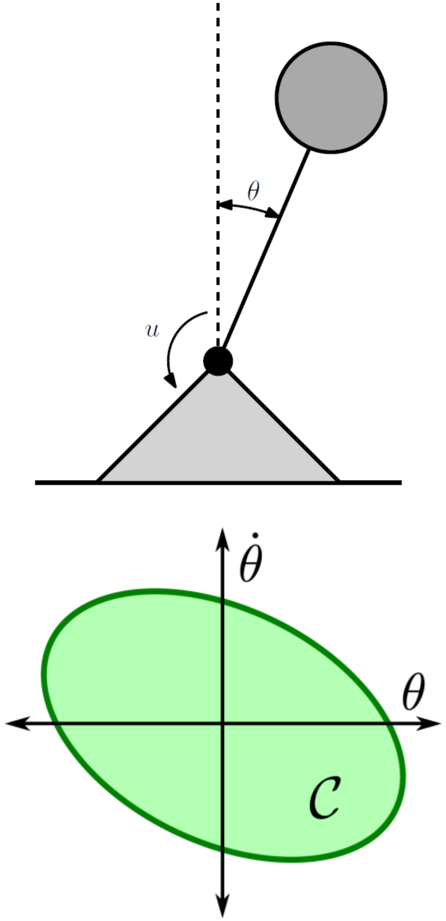




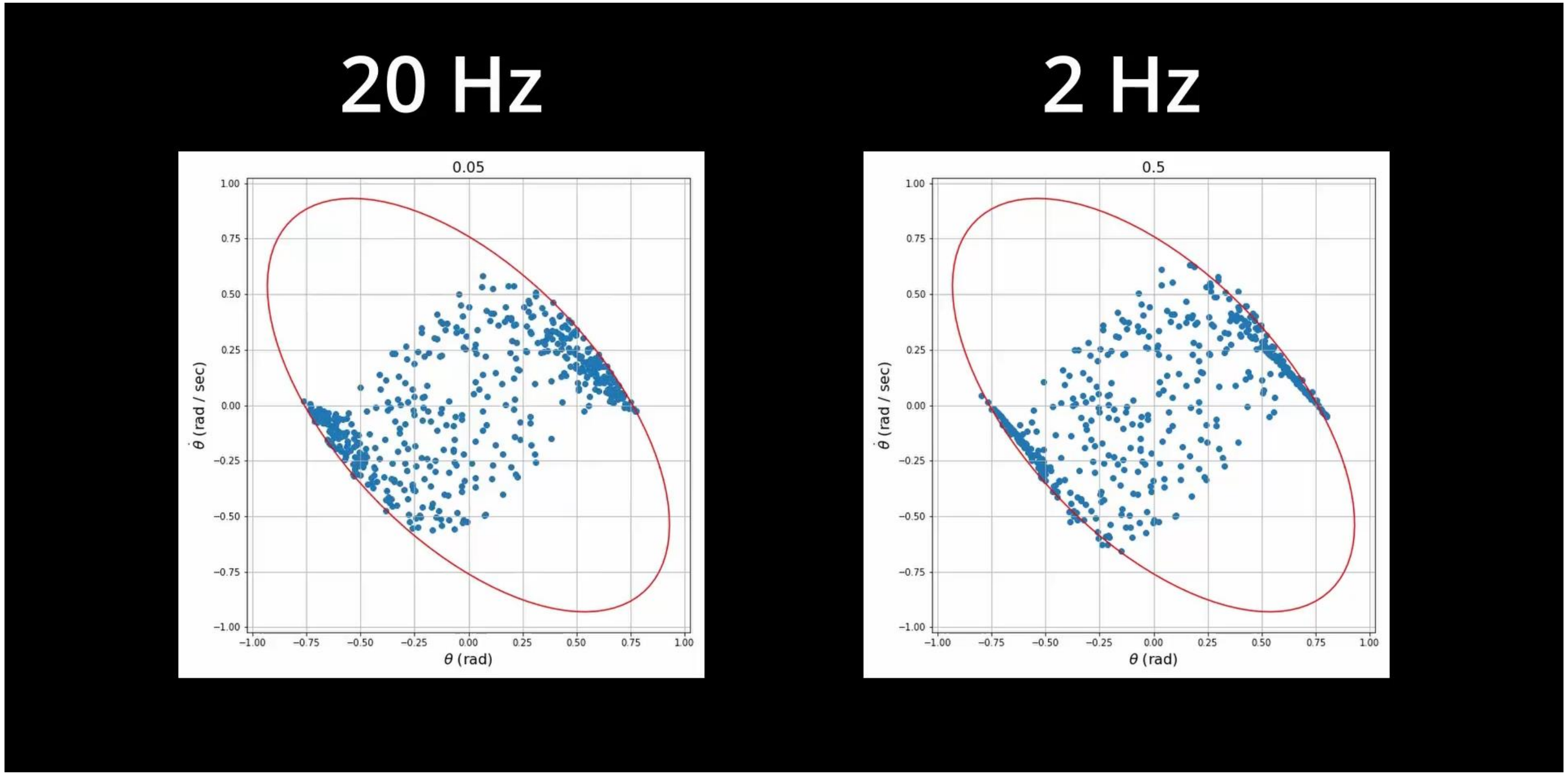
$$p = 2$$



Inverted Pendulum

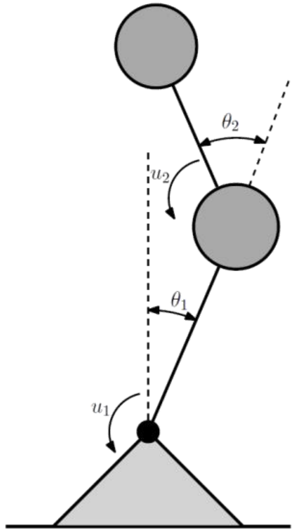


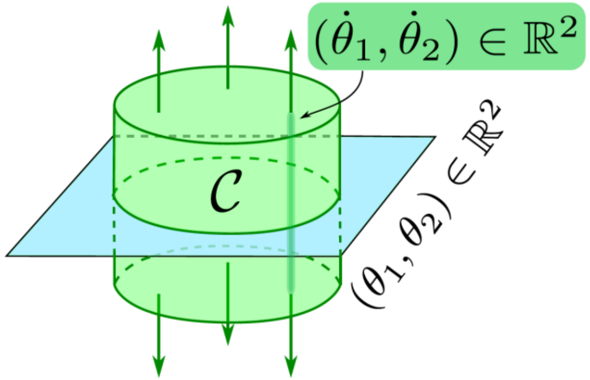
$$p = 1$$



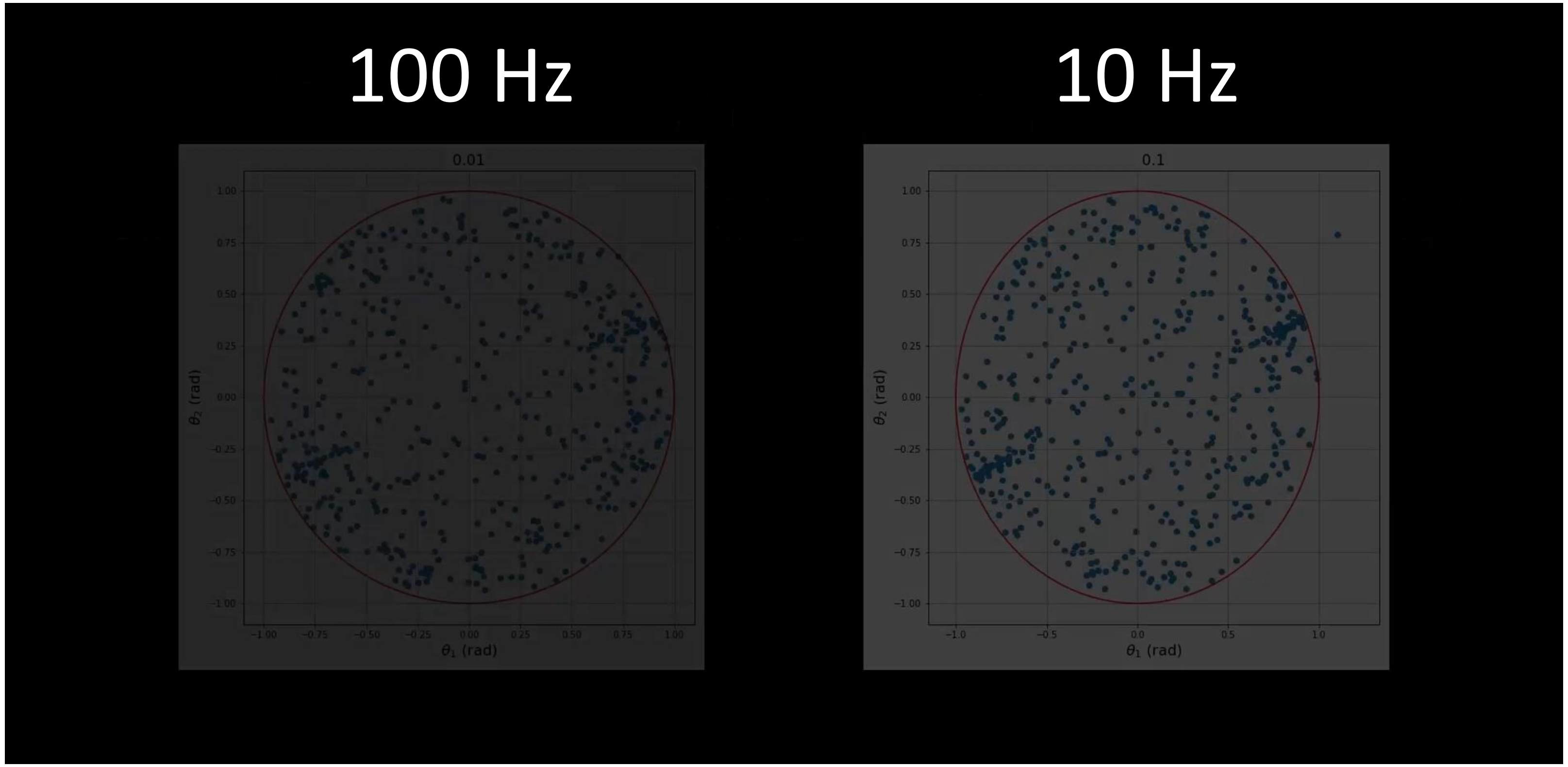
Double Inverted Pendulum

Caltech



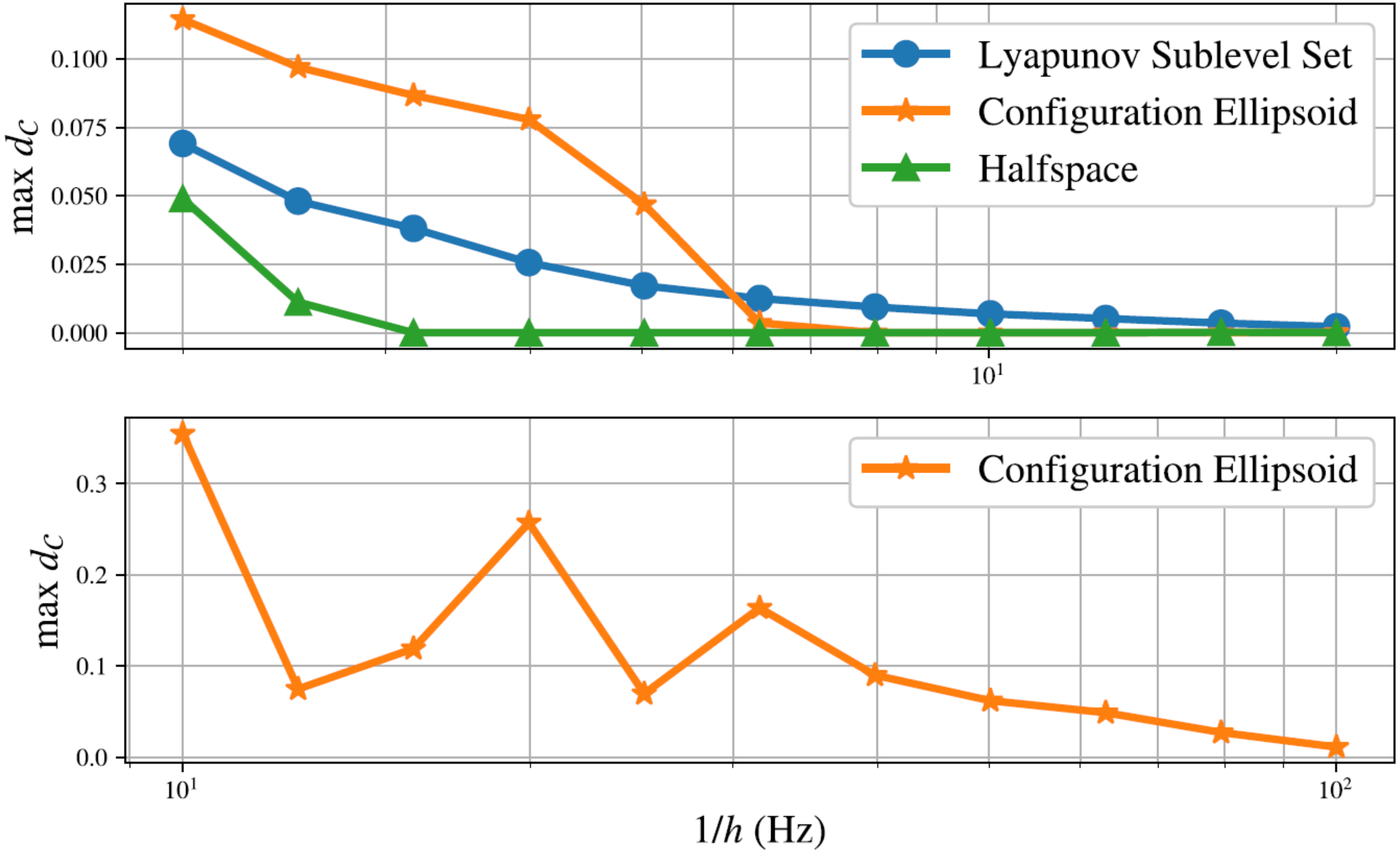


$$p = 2$$



Safety Violations

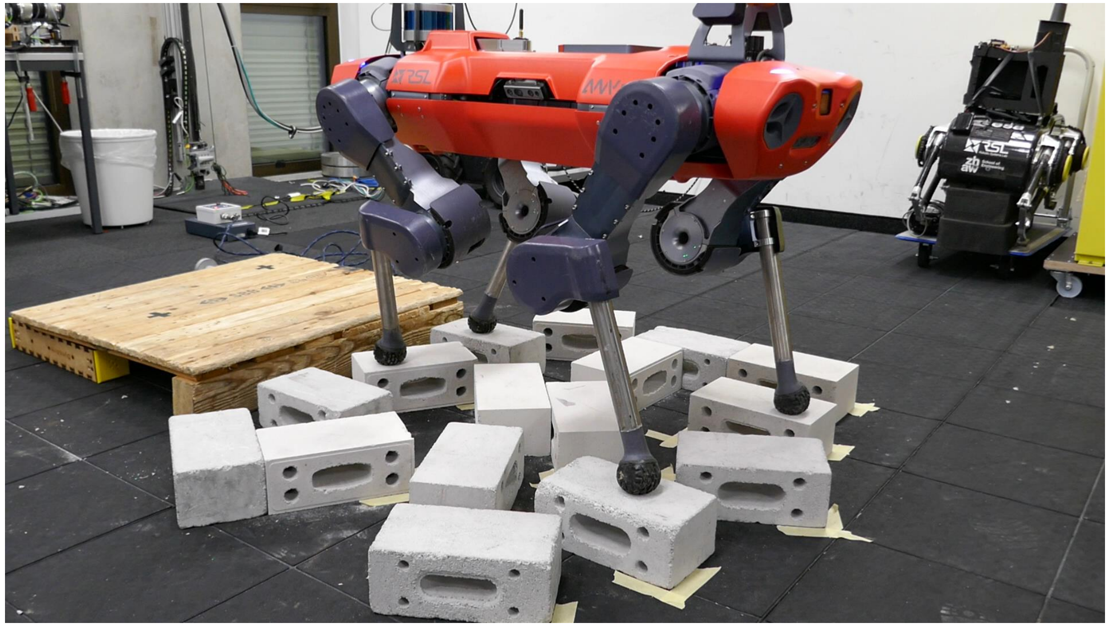
Caltech



Conclusions

Caltech

- Framework for achieving safety of sampled-data systems via **Control Barrier Functions (CBFs)** and **approximate discrete time models**
- Definition of **practical safety** analogous to that of practical stability for sampled-data systems
- Analysis of relationship between a CBF and an approximate discrete time model that yields **convex optimization-based** controllers



Caltech

Thank You!

Safety of Sampled-Data Systems with Control Barrier Functions via Approximate Discrete Time Models

Andrew J. Taylor Victor D. Dorobantu Ryan K. Cosner
Yisong Yue Aaron D. Ames