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Safe Backstepping with Control Barrier Functions

Andrew J. Taylor   Pio Ong   Tamas G. Molnár
Aaron D. Ames

Computing and Mathematical Sciences
California Institute of Technology

December 8th, 2022
Control & Decision Conference (CDC) 2022

Control for complex systems is hard

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But: Pretty when it works…

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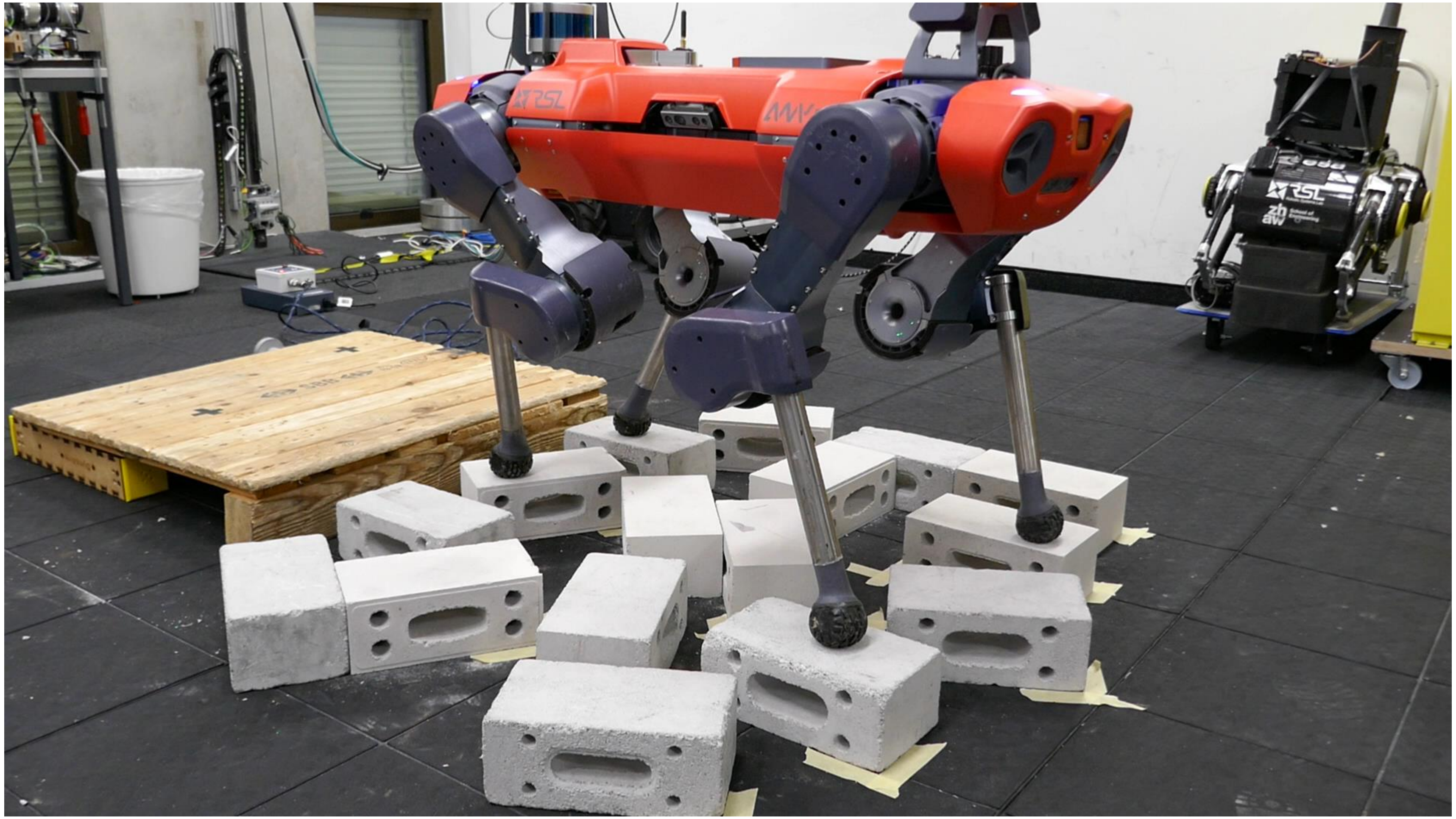

[1] R. Grandia, **A. J. Taylor**, M. Hutter, A. D. Ames, “Multi-Layered Safety for Legged Robotics via Control Barrier Functions and Model Predictive Control”, 2020.

Claim: Need to build constructive design tools

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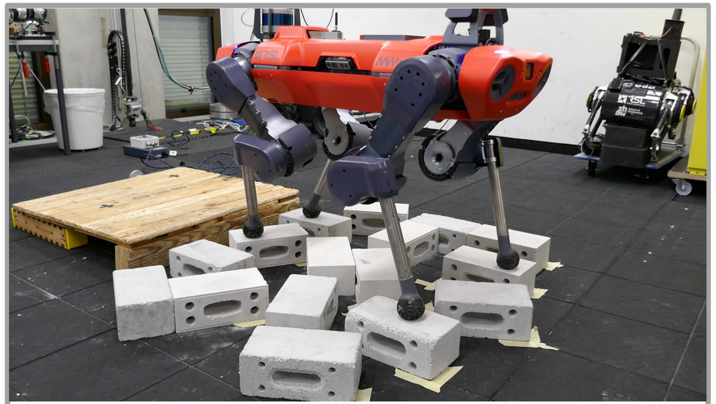

$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\mathrm{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t. } \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x}))$$

Theorems & Proofs

Bridge the Gap

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}$$
$$\dot{\boldsymbol{\xi}} = \mathbf{f}_1(\mathbf{x},\boldsymbol{\xi}) + \mathbf{g}_1(\mathbf{x},\boldsymbol{\xi})\mathbf{u}$$

Experimental Realization

Contributions

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- Framework for achieving safety of higher-order systems by unifying classical **Lyapunov backstepping** with **Control Barrier Functions**
- Constructive tool for **synthesizing** Control Barrier Functions for higher-order systems
- Design of **stable and safe** nonlinear controllers through joint Lyapunov and Barrier backstepping

System Dynamics

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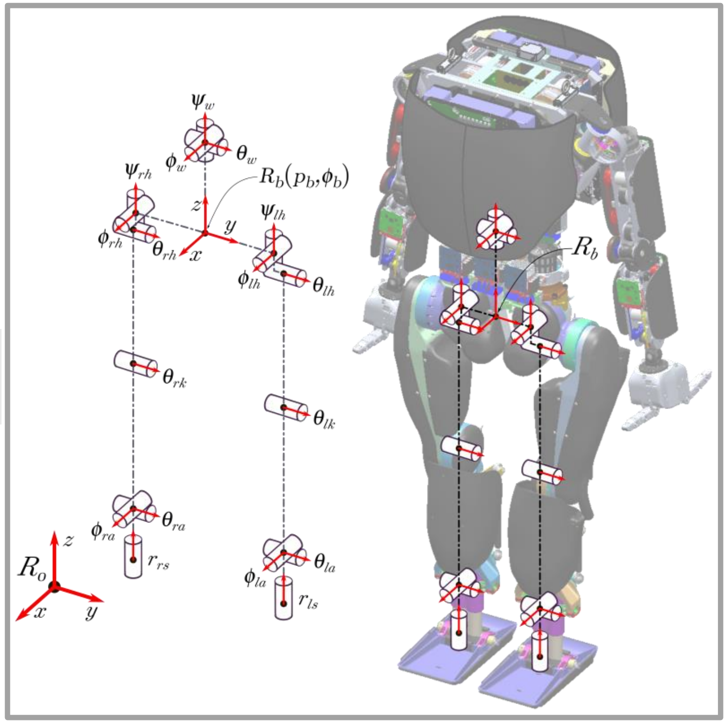
Equations of Motion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \qquad \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^{n \times m}$$

Mathematical Model



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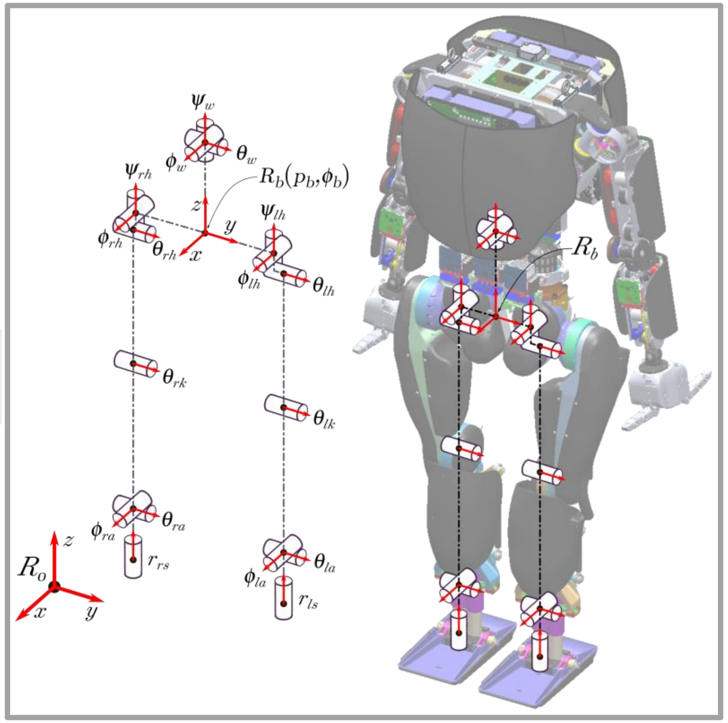
Assumptions

$\mathbf{f}$, $\mathbf{g}$ locally Lipschitz continuous

Mathematical Model



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Closed-Loop Solutions

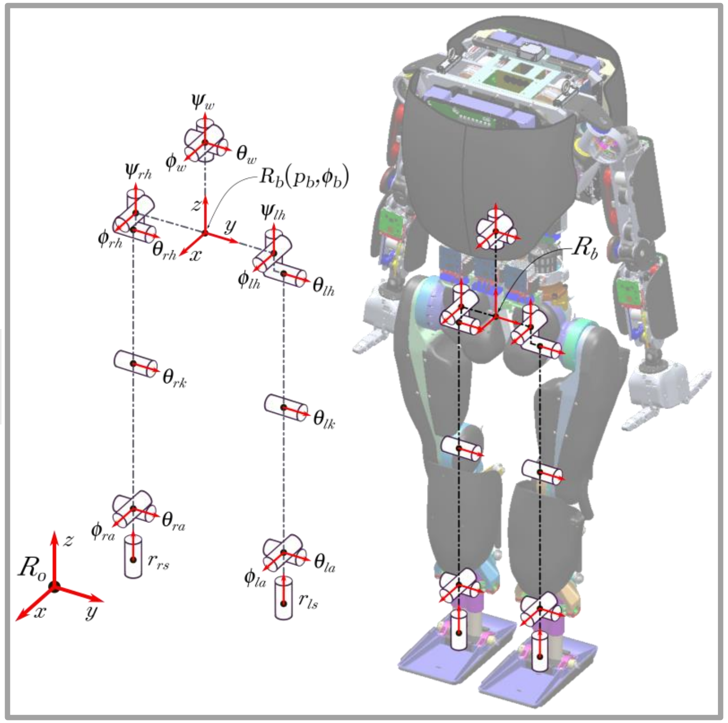
$$\mathbf{k}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$$

$$\mathbf{x}_0 \in \mathbb{R}^n \qquad \boldsymbol{\varphi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$$

$$\dot{\boldsymbol{\varphi}}(t) = \mathbf{f}(\boldsymbol{\varphi}(t)) + \mathbf{g}(\boldsymbol{\varphi}(t))\mathbf{k}(\boldsymbol{\varphi}(t))$$

$$\boldsymbol{\varphi}(0) = \mathbf{x}_0$$

Mathematical Model



System Model

Barrier Functions (BFs)

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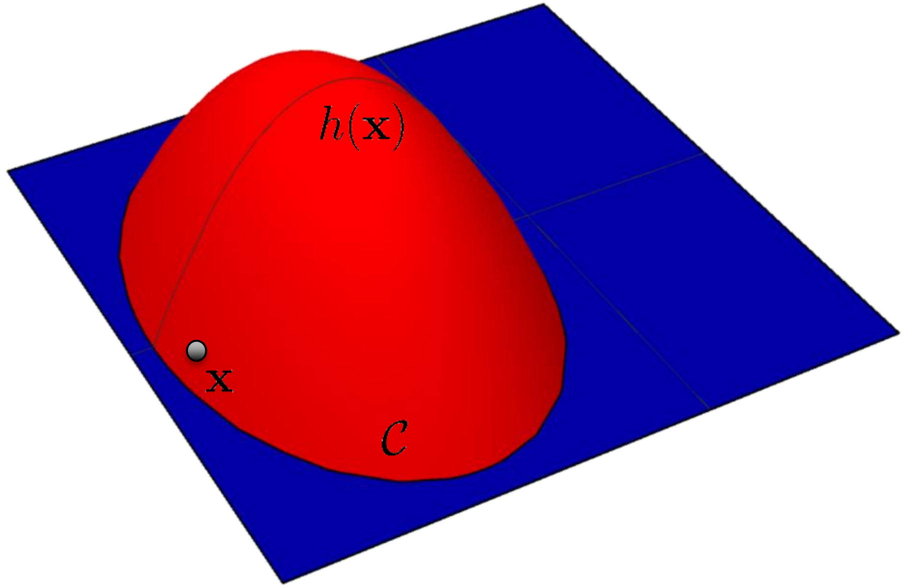
Safe Set

$$h : \mathbb{R}^n \to \mathbb{R}$$

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid h(\mathbf{x}) \geq 0\}$$



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Safe Set

$$h : \mathbb{R}^n \to \mathbb{R}$$
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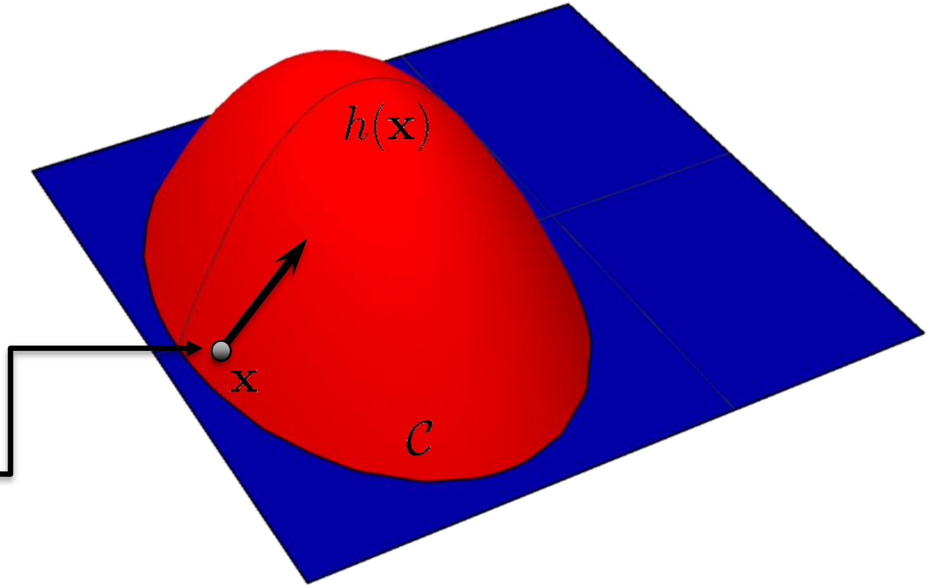
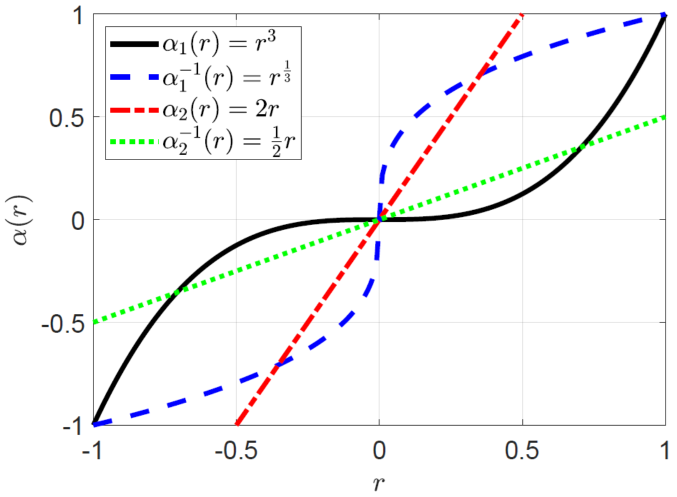
Barrier Function [2]

$$\dot{h}(\mathbf{x}, \mathbf{k}(\mathbf{x})) \geq -\alpha(h(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^n$$
$$\dot{h}(\mathbf{x}, \mathbf{u}) = \underbrace{\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}(\mathbf{x})}_{L_{\mathbf{f}}h(\mathbf{x})} + \underbrace{\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{g}(\mathbf{x})}_{L_{\mathbf{g}}h(\mathbf{x})} \mathbf{u}$$
$$\alpha \in \mathcal{K}_\infty^{\mathrm{e}}$$





[2] A. Ames, X. Xu, J. Grizzle, P. Tabuada, "Control Barrier Function Based Quadratic Programs for Safety Critical Systems", 2017.

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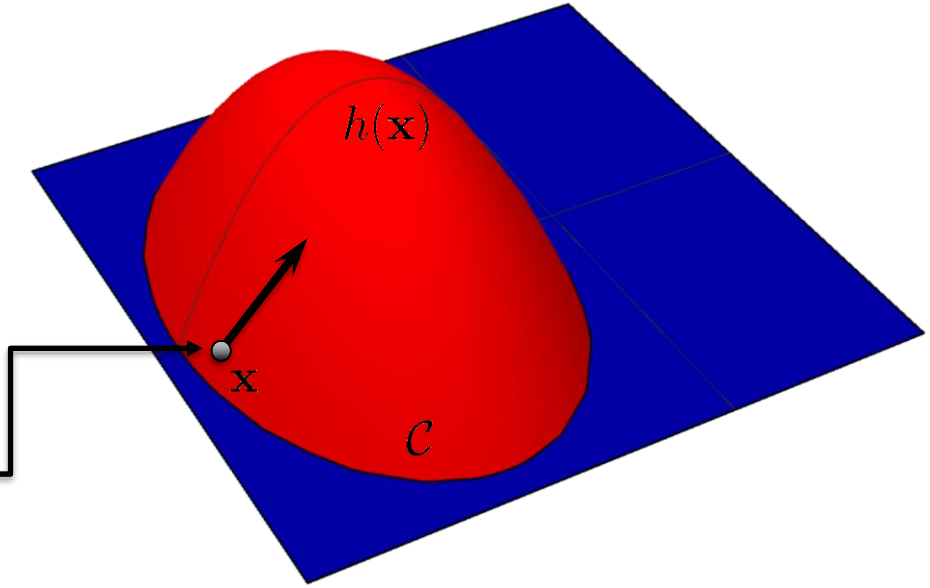
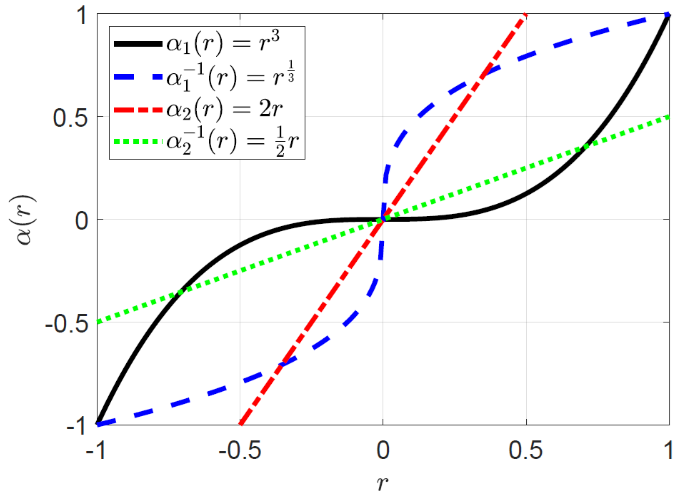
$$\alpha \in \mathcal{K}_\infty^{\mathrm{e}}$$

Safety [2]

$$\dot{h}(\mathbf{x}, \mathbf{k}(\mathbf{x})) \geq -\alpha(h(\mathbf{x})) \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$
$$\implies \mathcal{C} \text{ is forward invariant}$$





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Control Barrier Functions (CBFs)

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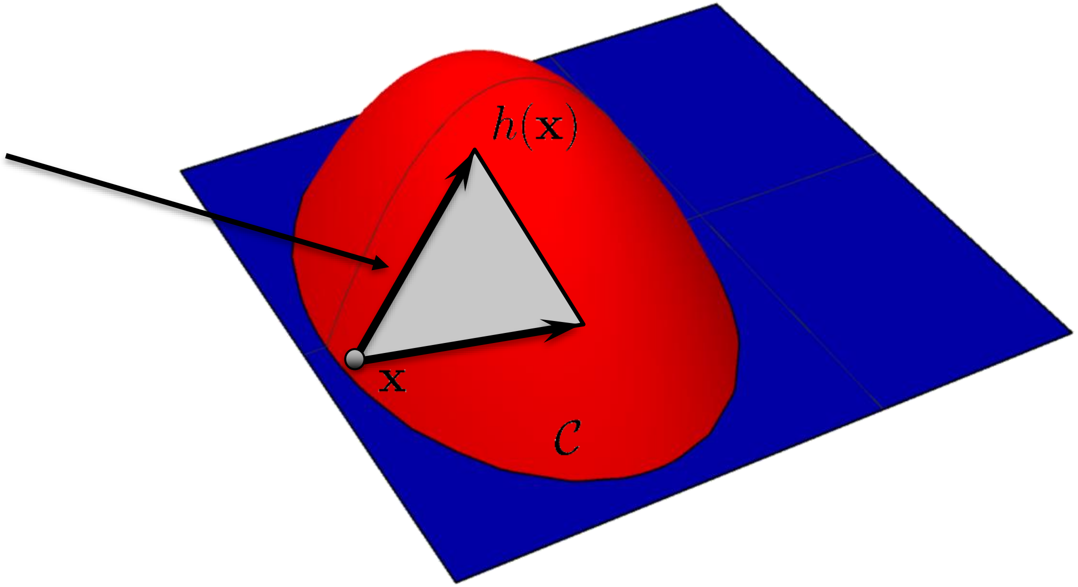
Control Barrier Function [2]

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}(\mathbf{x}, \mathbf{u}) > -\alpha(h(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^n$$



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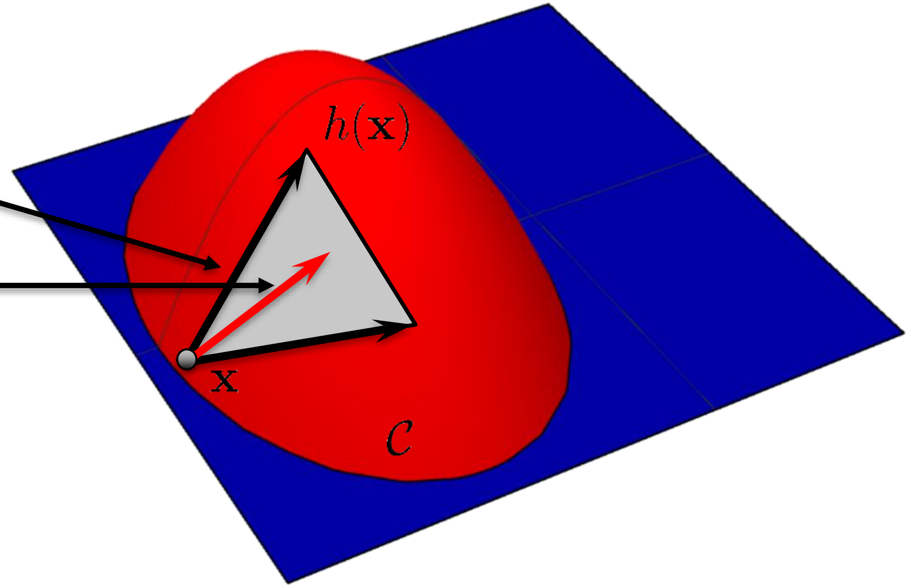
CBF Quadratic Program [2]

$$\mathbf{k}(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{u}\in\mathbb{R}^m} \|\mathbf{u}-\mathbf{k}_{\text{nom}}(\mathbf{x})\|_2^2$$
$$\text{s.t.} \quad \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x}))$$



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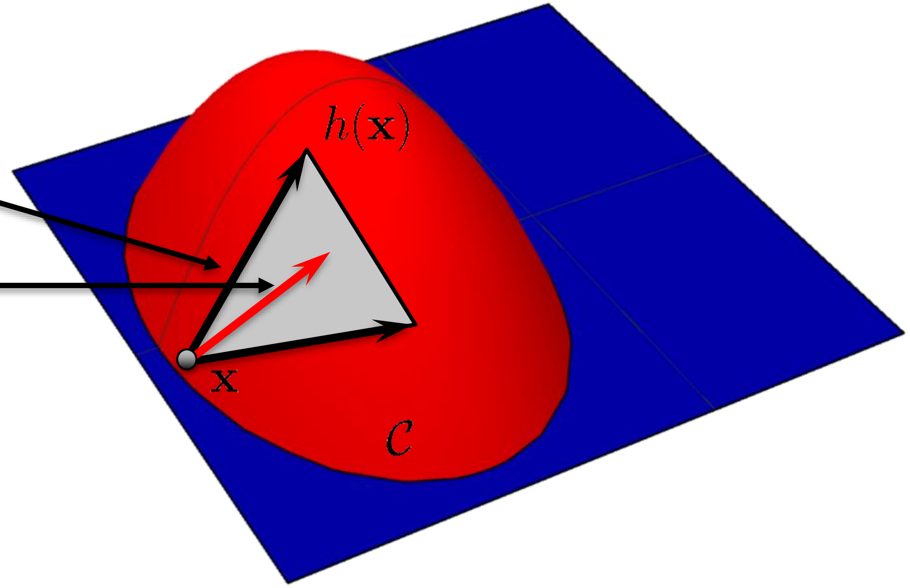
$$\begin{aligned}\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} & \ \|\mathbf{u}-\mathbf{k}_{\mathrm{nom}}(\mathbf{x})\|_2^2 \\ \text{s.t.} & \ \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x}))\end{aligned}$$



How do we work with higher-order systems?

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Cascaded Systems

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Single Cascade System

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}$$
$$\dot{\boldsymbol{\xi}} = \mathbf{f}_1(\mathbf{x}, \boldsymbol{\xi}) + \mathbf{g}_1(\mathbf{x}, \boldsymbol{\xi})\mathbf{u}$$

$\mathbf{x} \in \mathbb{R}^n$ $\boldsymbol{\xi} \in \mathbb{R}^p$ $\mathbf{u} \in \mathbb{R}^m$

$\mathbf{f}_0 : \mathbb{R}^n \to \mathbb{R}^n$ $\mathbf{g}_0 : \mathbb{R}^n \to \mathbb{R}^{n \times p}$

$\mathbf{f}_1 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ $\mathbf{g}_1 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{p \times m}$

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$\mathbf{f}_1 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ $\mathbf{g}_1 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{p\times m}$

Assumptions

$\mathbf{f}_0, \mathbf{g}_0, \mathbf{f}_1, \mathbf{g}_1$ locally Lipschitz continuous

$\mathbf{g}_1(\mathbf{x},\boldsymbol{\xi})$ pseudo-invertible for each $(\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^p$

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Top-Level Safe Set

$$\mathcal{C}_0 = \{\mathbf{x} \in \mathbb{R}^n \mid h_0(\mathbf{x}) \geq 0\}$$

$h_0 : \mathbb{R}^n \to \mathbb{R}$, twice continuously differentiable

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$$\mathcal{C}_0 = \{\mathbf{x} \in \mathbb{R}^n \mid h_0(\mathbf{x}) \geq 0\}$$

$h_0 : \mathbb{R}^n \to \mathbb{R}$, twice continuously differentiable

Barrier Derivative

$$\dot{h}_0(\mathbf{x}, \boldsymbol{\xi}) = \frac{\partial h_0}{\partial \mathbf{x}}(\mathbf{x}) \left(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}\right)$$

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Relative Degree Two

No control to ensure $\dot{h}_0(\mathbf{x},\boldsymbol{\xi}) \geq -\alpha_0(h_0(\mathbf{x}))$

$\alpha_0 \in \mathcal{K}_\infty^{\mathrm{e}},$ continuously differentiable

High-Order Control Barrier Functions

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High-Order Control Barrier Functions

[3] Q. Nguyen, K. Sreenath, “Exponential Control Barrier Functions for Enforcing High Relative-Degree Safety-Critical Constraints”, 2016.
[4] W. Xiao, C. Belta, “Control Barrier Functions for Systems with High Relative Degree”, 2021.
[5] W. Xiao, C. Belta, “High Order Control Barrier Functions”, 2021.
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Extended Barrier

$$h_1(\mathbf{x}, \boldsymbol{\xi}) = \dot{h}_0(\mathbf{x}, \boldsymbol{\xi}) + \alpha_0(h_0(\mathbf{x}))$$

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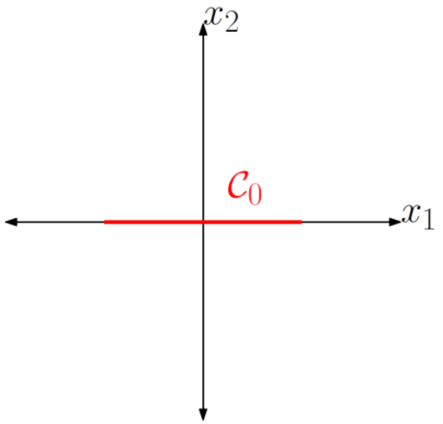

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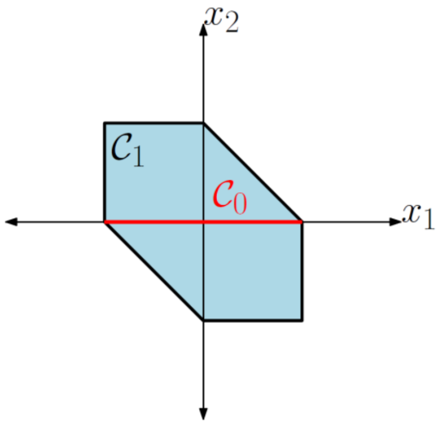

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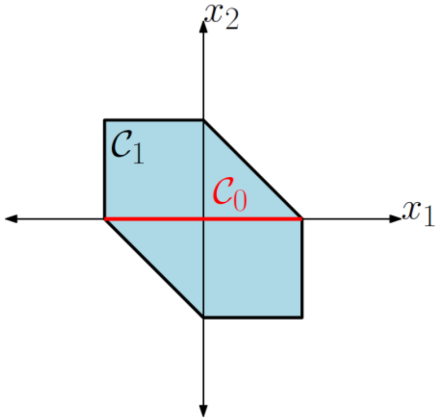

Barrier Time Derivative

$$\dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) = \frac{\partial h_1}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\xi}) \left(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}\right) + \frac{\partial h_1}{\partial \boldsymbol{\xi}}(\mathbf{x}, \boldsymbol{\xi}) \left(\mathbf{f}_1(\mathbf{x}, \boldsymbol{\xi}) + \mathbf{g}_1(\mathbf{x}, \boldsymbol{\xi})\mathbf{u}\right)$$

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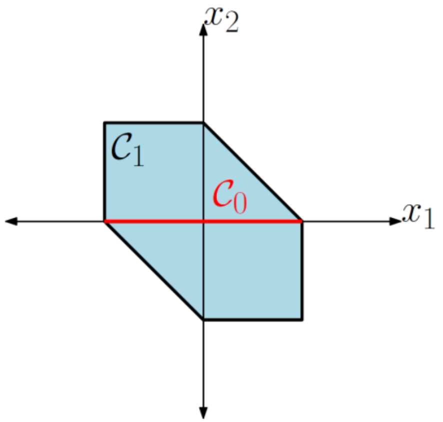

Barrier Time Derivative

$$\dot{h}_1(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) = \frac{\partial h_1}{\partial \mathbf{x}}(\mathbf{x},\boldsymbol{\xi})\left(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}\right) + \frac{\partial h_1}{\partial \boldsymbol{\xi}}(\mathbf{x},\boldsymbol{\xi})\left(\mathbf{f}_1(\mathbf{x},\boldsymbol{\xi}) + \mathbf{g}_1(\mathbf{x},\boldsymbol{\xi})\mathbf{u}\right)$$

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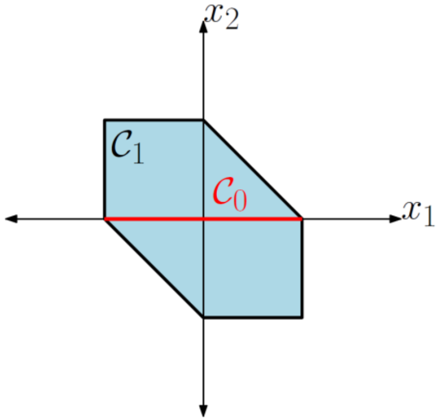
$$h_1(\mathbf{x},\boldsymbol{\xi}) = \dot{h}_0(\mathbf{x},\boldsymbol{\xi}) + \alpha_0(h_0(\mathbf{x}))$$

$$\mathcal{C}_1 = \{(\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^p \mid h_1(\mathbf{x},\boldsymbol{\xi}) \geq 0\}$$



Barrier Time Derivative

$$\dot{h}_1(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) = \frac{\partial h_1}{\partial \mathbf{x}}(\mathbf{x},\boldsymbol{\xi})\left(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}\right) + \frac{\partial h_1}{\partial \boldsymbol{\xi}}(\mathbf{x},\boldsymbol{\xi})\left(\mathbf{f}_1(\mathbf{x},\boldsymbol{\xi}) + \mathbf{g}_1(\mathbf{x},\boldsymbol{\xi})\mathbf{u}\right)$$

Controlled Safety

$$\dot{h}_1(\mathbf{x},\boldsymbol{\xi},\mathbf{k}(\mathbf{x},\boldsymbol{\xi})) \geq -\alpha_1(h_1(\mathbf{x},\boldsymbol{\xi}))$$
$$\implies \mathcal{C}_1 \text{ forward invariant}$$
$$\implies (\mathcal{C}_0 \times \mathbb{R}^p) \cap \mathcal{C}_1 \text{ forward invariant}$$

High-Order Control Barrier Functions

Caltech

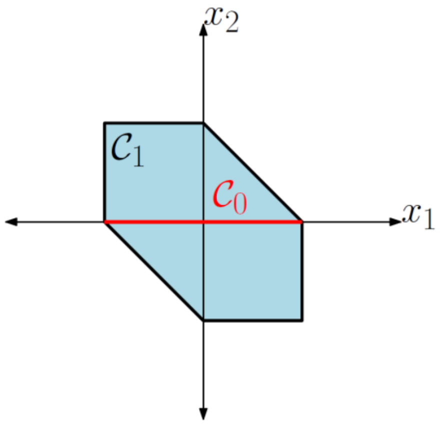
High-Order Control Barrier Functions

[3] Q. Nguyen, K. Sreenath, "Exponential Control Barrier Functions for Enforcing High Relative-Degree Safety-Critical Constraints", 2016.
[4] W. Xiao, C. Belta, "Control Barrier Functions for Systems with High Relative Degree", 2021.
[5] W. Xiao, C. Belta, "High Order Control Barrier Functions", 2021.
[6] J. Breeden, D. Panagou, "High Relative Degree Control Barrier Functions Under Input Constraints", 2021.

Extended Barrier

$$h_1(\mathbf{x},\boldsymbol{\xi}) = \dot{h}_0(\mathbf{x},\boldsymbol{\xi}) + \alpha_0(h_0(\mathbf{x}))$$

$$\mathcal{C}_1 = \{(\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^p \mid h_1(\mathbf{x},\boldsymbol{\xi}) \geq 0\}$$



Barrier Time Derivative

$$\dot{h}_1(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) = \frac{\partial h_1}{\partial \mathbf{x}}(\mathbf{x},\boldsymbol{\xi})\left(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}\right) + \frac{\partial h_1}{\partial \boldsymbol{\xi}}(\mathbf{x},\boldsymbol{\xi})\left(\mathbf{f}_1(\mathbf{x},\boldsymbol{\xi}) + \mathbf{g}_1(\mathbf{x},\boldsymbol{\xi})\mathbf{u}\right)$$

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$$\implies \mathcal{C}_1 \text{ forward invariant}$$
$$\implies (\mathcal{C}_0 \times \mathbb{R}^p) \cap \mathcal{C}_1 \text{ forward invariant}$$

Optimization-Based Control

$$\mathbf{k}(\mathbf{x},\boldsymbol{\xi}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\text{nom}}(\mathbf{x},\boldsymbol{\xi})\|_2^2$$
$$\text{s.t. } \dot{h}_1(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) \geq -\alpha_1(h_1(\mathbf{x},\boldsymbol{\xi}))$$

High-Order Control Barrier Functions

Caltech

Control Barrier Function Condition

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) > -\alpha_1(h_1(\mathbf{x}, \boldsymbol{\xi}))$$

High-Order Control Barrier Functions

Caltech

Control Barrier Function Condition

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) > -\alpha_1(h_1(\mathbf{x}, \boldsymbol{\xi}))$$

Is this satisfied?

High-Order Control Barrier Functions

Caltech

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Is this satisfied?

Relative Degree Assumption

$$\frac{\partial h_1}{\partial \boldsymbol{\xi}}(\mathbf{x},\boldsymbol{\xi})\mathbf{g}_1(\mathbf{x},\boldsymbol{\xi}) \neq \mathbf{0} \quad \text{for all } (\mathbf{x},\boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^p$$

High-Order Control Barrier Functions

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Control Barrier Function Condition

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) > -\alpha_1(h_1(\mathbf{x}, \boldsymbol{\xi}))$$

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Not true in many interesting cases

High-Order Control Barrier Functions

Caltech

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$$\sup_{\mathbf{u}\in\mathbb{R}^m} \dot{h}_1(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) > -\alpha_1(h_1(\mathbf{x},\boldsymbol{\xi}))$$

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Not true in many interesting cases

Alternative CBF Condition

$$\frac{\partial h_1}{\partial \boldsymbol{\xi}}(\mathbf{x},\boldsymbol{\xi})\mathbf{g}_1(\mathbf{x},\boldsymbol{\xi}) = \mathbf{0} \implies \frac{\partial h_1}{\partial \mathbf{x}}(\mathbf{x},\boldsymbol{\xi})\left(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}\right) + \frac{\partial h_1}{\partial \boldsymbol{\xi}}(\mathbf{x},\boldsymbol{\xi})\mathbf{f}_1(\mathbf{x},\boldsymbol{\xi}) > -\alpha_1(h_1(\mathbf{x},\boldsymbol{\xi}))$$

High-Order Control Barrier Functions

Caltech

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$$\sup_{\mathbf{u}\in\mathbb{R}^m} \dot{h}_1(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) > -\alpha_1(h_1(\mathbf{x},\boldsymbol{\xi}))$$

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Difficult to check in general

High-Order Control Barrier Functions

Can we do something more constructive?

Caltech

Control Barrier Function Condition

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) > -\alpha_1(h_1(\mathbf{x}, \boldsymbol{\xi}))$$

← Is this satisfied?

Relative Degree Assumption

$$\frac{\partial h_1}{\partial \boldsymbol{\xi}}(\mathbf{x}, \boldsymbol{\xi})\mathbf{g}_1(\mathbf{x}, \boldsymbol{\xi}) \neq \mathbf{0} \quad \text{for all } (\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^p$$

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← Difficult to check in general

Reduced-Order Model Design

Caltech

Reduced-Order Controller[7]

$$\mathbf{k}_0 : \mathbb{R}^n \to \mathbb{R}^p$$
$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$

[7] T. Molnár, R. Cosner, A. Singletary, W. Ubellacker, A. Ames, "Model-Free Safety-Critical Control for Robotic Systems", 2022.

Reduced-Order Model Design

Caltech

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Full-Order Controller[7]

$$\mathbf{k}_1 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m \quad V : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$$
$$c_1 \|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|_2^2 \leq V(\mathbf{x}, \boldsymbol{\xi}) \leq c_2 \|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|_2^2$$
$$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}_1(\mathbf{x}, \boldsymbol{\xi})) \leq -c_3 V(\mathbf{x}, \boldsymbol{\xi})$$

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$$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}_1(\mathbf{x}, \boldsymbol{\xi})) \leq -c_3 V(\mathbf{x}, \boldsymbol{\xi})$$

Composite Barrier Function[7]

$$h_1(\mathbf{x}, \boldsymbol{\xi}) = h_0(\mathbf{x}) - \lambda V(\mathbf{x}, \boldsymbol{\xi})$$
$$\lambda \in \mathbb{R}_{>0}$$

[7] T. Molnár, R. Cosner, A. Singletary, W. Ubellacker, A. Ames, "Model-Free Safety-Critical Control for Robotic Systems", 2022.

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Caltech

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Safety[7]

$$\dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}_1(\mathbf{x}, \boldsymbol{\xi})) \geq -\alpha_1 h_1(\mathbf{x}, \boldsymbol{\xi})$$
$$\implies h_1(\boldsymbol{\varphi}_{\mathbf{x}}(t), \boldsymbol{\varphi}_{\boldsymbol{\xi}}(t)) \geq 0$$
$$\implies h_0(\boldsymbol{\varphi}_{\mathbf{x}}(t)) \geq V(\boldsymbol{\varphi}_{\mathbf{x}}(t), \boldsymbol{\varphi}_{\boldsymbol{\xi}}(t)) \geq 0$$

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Reduced-Order Model Design

Caltech

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$$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}_1(\mathbf{x}, \boldsymbol{\xi})) \leq -c_3 V(\mathbf{x}, \boldsymbol{\xi})$$

How do we do this?

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Caltech

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$$\implies h_0(\boldsymbol{\varphi}_{\mathbf{x}}(t)) \geq V(\boldsymbol{\varphi}_{\mathbf{x}}(t), \boldsymbol{\varphi}_{\boldsymbol{\xi}}(t)) \geq 0$$

How do we do this?

Does this look like backstepping?

[7] T. Molnár, R. Cosner, A. Singletary, W. Ubellacker, A. Ames, "Model-Free Safety-Critical Control for Robotic Systems", 2022.

Lyapunov Backstepping

Caltech

Equilibrium Point

$$\mathbf{f}_0(\mathbf{0}) = \mathbf{0} \qquad \mathbf{f}_1(\mathbf{0}, \mathbf{0}) = \mathbf{0}$$

Lyapunov Backstepping

Caltech

Equilibrium Point

$$\mathbf{f}_0(\mathbf{0}) = \mathbf{0} \qquad \mathbf{f}_1(\mathbf{0},\mathbf{0}) = \mathbf{0}$$

Top-Level Design

$\mathbf{k}_0 : \mathbb{R}^n \to \mathbb{R}^p,$ twice continuously differentiable

$V_0 : \mathbb{R}^n \to \mathbb{R}_{\geq 0},$ twice continuously differentiable

$$\gamma_1(\|\mathbf{x}\|) \leq V_0(\mathbf{x}) \leq \gamma_2(\|\mathbf{x}\|)$$

$$\frac{\partial V_0}{\partial \mathbf{x}}(\mathbf{x})\left(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{k}_0(\mathbf{x})\right) \leq -\gamma_3(\|\mathbf{x}\|)$$

$$\mathbf{k}_0(\mathbf{0}) = \mathbf{0} \qquad \gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$$

Lyapunov Backstepping

Caltech

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$$\mathbf{k}_0(\mathbf{0}) = \mathbf{0} \qquad \gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$$

Composite Lyapunov Function

$$V(\mathbf{x}, \boldsymbol{\xi}) = V_0(\mathbf{x}) + \frac{1}{2\mu}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$

$$\mu \in \mathbb{R}_{>0}$$

Lyapunov Backstepping

Caltech

Equilibrium Point

$$\mathbf{f}_0(\mathbf{0}) = \mathbf{0} \qquad \mathbf{f}_1(\mathbf{0},\mathbf{0}) = \mathbf{0}$$

Top-Level Design

$$\mathbf{k}_0 : \mathbb{R}^n \to \mathbb{R}^p, \ \text{twice continuously differentiable}$$

$$V_0 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \ \text{twice continuously differentiable}$$

$$\gamma_1(\|\mathbf{x}\|) \leq V_0(\mathbf{x}) \leq \gamma_2(\|\mathbf{x}\|)$$

$$\frac{\partial V_0}{\partial \mathbf{x}}(\mathbf{x})\left(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{k}_0(\mathbf{x})\right) \leq -\gamma_3(\|\mathbf{x}\|)$$

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$$\mu \in \mathbb{R}_{>0}$$

Structured Low-Level Controller

$$\mathbf{k}(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{g}_1(\mathbf{x}, \boldsymbol{\xi})^\dagger \Bigg(-\mathbf{f}_1(\mathbf{x}, \boldsymbol{\xi}) + \frac{\partial \mathbf{k}_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}) - \mu\left(\frac{\partial V_0}{\partial \mathbf{x}}(\mathbf{x})\mathbf{g}_0(\mathbf{x})\right)^\top - \frac{\lambda}{2}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))\Bigg)$$

$$\lambda \in \mathbb{R}_{>0}$$

Lyapunov Backstepping

Caltech

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$$\lambda \in \mathbb{R}_{>0}$$

Lyapunov Decay

$$\dot{V}(\mathbf{x},\boldsymbol{\xi},\mathbf{k}(\mathbf{x},\boldsymbol{\xi})) \leq -\gamma_3(\|\mathbf{x}\|) - \gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|)$$

Lyapunov Backstepping

Caltech

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$$\mu \in \mathbb{R}_{>0}$$

Structured Low-Level Controller

$$\mathbf{k}(\mathbf{x},\boldsymbol{\xi}) = \mathbf{g}_1(\mathbf{x},\boldsymbol{\xi})^\dagger \left(-\mathbf{f}_1(\mathbf{x},\boldsymbol{\xi}) + \frac{\partial \mathbf{k}_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}) - \mu\left(\frac{\partial V_0}{\partial \mathbf{x}}(\mathbf{x})\mathbf{g}_0(\mathbf{x})\right)^\top - \frac{\lambda}{2}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))\right)$$

$$\lambda \in \mathbb{R}_{>0}$$

Lyapunov Decay

$$\dot{V}(\mathbf{x},\boldsymbol{\xi},\mathbf{k}(\mathbf{x},\boldsymbol{\xi})) \leq -\gamma_3(\|\mathbf{x}\|) - \gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|)$$

Asymptotic Stability

$$\begin{aligned} \boldsymbol{\varphi}_{\mathbf{x}}(t) &\to \mathbf{0} \\ \boldsymbol{\varphi}_{\boldsymbol{\xi}}(t) - \mathbf{k}_0(\boldsymbol{\varphi}_{\mathbf{x}}(t)) &\to \mathbf{0} \end{aligned} \quad \text{as } t \to \infty$$

Lyapunov Backstepping

Caltech

Lyapunov Decay
$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}(\mathbf{x}, \boldsymbol{\xi})) \leq -\gamma_3(\|\mathbf{x}\|) - \gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|)$

Lyapunov Backstepping

Caltech

Lyapunov Decay
$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}(\mathbf{x}, \boldsymbol{\xi})) \leq -\gamma_3(\|\mathbf{x}\|) - \gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|)$

→

Strict Lyapunov Condition
$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}(\mathbf{x}, \boldsymbol{\xi})) < -c\gamma_3(\|\mathbf{x}\|) - c\gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|)$ $c \in (0, 1)$

Lyapunov Backstepping

Caltech

Lyapunov Decay

$$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}(\mathbf{x}, \boldsymbol{\xi})) \leq -\gamma_3(\|\mathbf{x}\|) - \gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|)$$

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$$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}(\mathbf{x}, \boldsymbol{\xi})) < -c\gamma_3(\|\mathbf{x}\|) - c\gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|)$$

$$c \in (0, 1)$$

Control Lyapunov Function

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) < -c\gamma_3(\|\mathbf{x}\|) - c\gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|)$$

Lyapunov Backstepping

Caltech

Lyapunov Decay

$$\dot{V}(\mathbf{x},\boldsymbol{\xi},\mathbf{k}(\mathbf{x},\boldsymbol{\xi})) \leq -\gamma_3(\|\mathbf{x}\|) - \gamma_3'(\|\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x})\|)$$

Strict Lyapunov Condition

$$\dot{V}(\mathbf{x},\boldsymbol{\xi},\mathbf{k}(\mathbf{x},\boldsymbol{\xi})) < -c\gamma_3(\|\mathbf{x}\|) - c\gamma_3'(\|\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x})\|)$$

$$c \in (0,1)$$

Control Lyapunov Function

$$\inf_{\mathbf{u}\in\mathbb{R}^m} \dot{V}(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) < -c\gamma_3(\|\mathbf{x}\|) - c\gamma_3'(\|\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x})\|)$$

Optimization-Based Control

$$\mathbf{k}(\mathbf{x},\boldsymbol{\xi}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2$$
$$\text{s.t. } \dot{V}(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) \leq -c\gamma_3(\|\mathbf{x}\|) - c\gamma_3'(\|\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x})\|)$$

Barrier Backstepping

Caltech

Top-Level Safe Set
$\mathcal{C}_0 = \{\mathbf{x} \in \mathbb{R}^n \mid h_0(\mathbf{x}) \geq 0\}$

Reduced-Order Controller
$\mathbf{k}_0 : \mathbb{R}^n \to \mathbb{R}^p$ $\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$ α_0 globally Lipschitz

Barrier Backstepping

Caltech

Top-Level Safe Set

$$\mathcal{C}_0 = \{\mathbf{x} \in \mathbb{R}^n \mid h_0(\mathbf{x}) \geq 0\}$$

Reduced-Order Controller

$$\mathbf{k}_0 : \mathbb{R}^n \to \mathbb{R}^p$$
$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$
α_0 globally Lipschitz

Composite Barrier Function

$$h_1(\mathbf{x}, \boldsymbol{\xi}) = h_0(\mathbf{x}) - \frac{1}{2\mu}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$
$$\mu \in \mathbb{R}_{>0}$$

Barrier Backstepping

Caltech

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$\mu \in \mathbb{R}_{>0}$

Composite Safe Set

$\mathcal{C}_1 = \{(\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^p \mid h_1(\mathbf{x}, \boldsymbol{\xi}) \geq 0\}$

Barrier Backstepping

Caltech

Top-Level Safe Set

$$\mathcal{C}_0 = \{\mathbf{x} \in \mathbb{R}^n \mid h_0(\mathbf{x}) \geq 0\}$$

Reduced-Order Controller

$$\mathbf{k}_0 : \mathbb{R}^n \to \mathbb{R}^p$$
$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$
$$\alpha_0 \text{ globally Lipschitz}$$

Set Containment

$$h_1(\mathbf{x}, \boldsymbol{\xi}) \geq 0 \implies h_0(\mathbf{x}) \geq 0$$
$$\mathcal{C}_1 \subseteq \mathcal{C}_0 \times \mathbb{R}^p$$

Composite Barrier Function

$$h_1(\mathbf{x}, \boldsymbol{\xi}) = h_0(\mathbf{x}) - \frac{1}{2\mu}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$
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Barrier Backstepping

Caltech

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$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$
$$\alpha_0 \text{ globally Lipschitz}$$

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$$h_1(\mathbf{x}, \boldsymbol{\xi}) \geq 0 \implies h_0(\mathbf{x}) \geq 0$$
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Composite Barrier Function

$$h_1(\mathbf{x}, \boldsymbol{\xi}) = h_0(\mathbf{x}) - \frac{1}{2\mu}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$
$$\mu \in \mathbb{R}_{>0}$$

Composite Safe Set

$$\mathcal{C}_1 = \{(\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^p \mid h_1(\mathbf{x}, \boldsymbol{\xi}) \geq 0\}$$

Structured Controller

$$\mathbf{k}(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{g}_1(\mathbf{x}, \boldsymbol{\xi})^\dagger \Big(- \mathbf{f}_1(\mathbf{x}, \boldsymbol{\xi}) + \frac{\partial \mathbf{k}_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}) + \mu\left(\frac{\partial h_0}{\partial \mathbf{x}}(\mathbf{x})\mathbf{g}_0(\mathbf{x})\right)^\top - \frac{\lambda}{2}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))\Big)$$
$$\lambda \geq \mathcal{L}_{\alpha_0}$$

Caltech

Barrier Backstepping

Top-Level Safe Set

$$\mathcal{C}_0 = \{\mathbf{x} \in \mathbb{R}^n \mid h_0(\mathbf{x}) \geq 0\}$$

Reduced-Order Controller

$$\mathbf{k}_0 : \mathbb{R}^n \to \mathbb{R}^p$$

$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$

α_0 globally Lipschitz

Set Containment

$$h_1(\mathbf{x}, \boldsymbol{\xi}) \geq 0 \implies h_0(\mathbf{x}) \geq 0$$

$$\mathcal{C}_1 \subseteq \mathcal{C}_0 \times \mathbb{R}^p$$

Composite Barrier Function

$$h_1(\mathbf{x}, \boldsymbol{\xi}) = h_0(\mathbf{x}) - \frac{1}{2\mu}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$

$$\mu \in \mathbb{R}_{>0}$$

Composite Safe Set

$$\mathcal{C}_1 = \{(\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^p \mid h_1(\mathbf{x}, \boldsymbol{\xi}) \geq 0\}$$

Structured Controller

$$\mathbf{k}(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{g}_1(\mathbf{x}, \boldsymbol{\xi})^\dagger \Big(-\mathbf{f}_1(\mathbf{x}, \boldsymbol{\xi}) + \frac{\partial \mathbf{k}_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\boldsymbol{\xi}) + \mu\left(\frac{\partial h_0}{\partial \mathbf{x}}(\mathbf{x})\mathbf{g}_0(\mathbf{x})\right)^\top - \frac{\lambda}{2}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))\Big)$$

$$\lambda \geq \mathcal{L}_{\alpha_0}$$

Safety

$$\dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}(\mathbf{x}, \boldsymbol{\xi})) \geq -\alpha_0(h_1(\mathbf{x}, \boldsymbol{\xi}))$$

$$h_1(\boldsymbol{\varphi}_{\mathbf{x}}(t), \boldsymbol{\varphi}_{\boldsymbol{\xi}}(t)) \geq 0 \implies h_0(\boldsymbol{\varphi}_{\mathbf{x}}(t)) \geq 0$$

Constructive Control Barrier Functions

Caltech

Strict Top-Level Barrier Function

$$\frac{\partial h_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{k}_0(\mathbf{x})) > -\alpha_0(h_0(\mathbf{x}))$$

Constructive Control Barrier Functions

Caltech

Strict Top-Level Barrier Function

$$\frac{\partial h_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{k}_0(\mathbf{x})) > -\alpha_0(h_0(\mathbf{x}))$$

↓

Full System CBF

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) > -\alpha_0(h_1(\mathbf{x}, \boldsymbol{\xi}))$$

Construct CBFs for complex systems using CBFs for simple systems!

Constructive Control Barrier Functions

Caltech

Strict Top-Level Barrier Function

$$\frac{\partial h_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{k}_0(\mathbf{x})) > -\alpha_0(h_0(\mathbf{x}))$$

⇓

Full System CBF

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) > -\alpha_0(h_1(\mathbf{x}, \boldsymbol{\xi}))$$

→

CBF-QP

$$\mathbf{k}(\mathbf{x}, \boldsymbol{\xi}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_{\mathrm{nom}}(\mathbf{x}, \boldsymbol{\xi})\|_2^2$$
$$\text{s.t. } \dot{h}_1(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) \geq -\alpha_0(h_1(\mathbf{x}, \boldsymbol{\xi}))$$

Construct CBFs for complex systems using CBFs for simple systems!

Do not need to use structured controller

Multi-Step Barrier Backstepping

Caltech

Multi-Cascade System

$$\begin{aligned}
\dot{\boldsymbol{\xi}}_0 &= \mathbf{f}_0(\boldsymbol{\xi}_0) + \mathbf{g}_{0,\boldsymbol{\xi}}(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 + \mathbf{g}_{0,\mathbf{u}}(\boldsymbol{\xi}_0)\mathbf{u}_0, \\
\dot{\boldsymbol{\xi}}_1 &= \mathbf{f}_1(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1) + \mathbf{g}_{1,\boldsymbol{\xi}}(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1)\boldsymbol{\xi}_2 + \mathbf{g}_{1,\mathbf{u}}(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1)\mathbf{u}_1, \\
&\vdots \\
\dot{\boldsymbol{\xi}}_r &= \mathbf{f}_r(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\ldots\boldsymbol{\xi}_r) + \mathbf{g}_r(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_r)\mathbf{u}_r,
\end{aligned}$$

$$\boldsymbol{\xi}_i \in \mathbb{R}^{p_i} \quad \mathbf{z}_i = (\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_i) \in \mathbb{R}^{q_i} \quad i = 0,\ldots,r$$

Multi-Step Barrier Backstepping

Caltech

Multi-Cascade System

$$\begin{aligned}
\dot{\boldsymbol{\xi}}_0 &= \mathbf{f}_0(\boldsymbol{\xi}_0) + \mathbf{g}_{0,\boldsymbol{\xi}}(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 + \mathbf{g}_{0,\mathbf{u}}(\boldsymbol{\xi}_0)\mathbf{u}_0, \\
\dot{\boldsymbol{\xi}}_1 &= \mathbf{f}_1(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1) + \mathbf{g}_{1,\boldsymbol{\xi}}(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1)\boldsymbol{\xi}_2 + \mathbf{g}_{1,\mathbf{u}}(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1)\mathbf{u}_1, \\
&\vdots \\
\dot{\boldsymbol{\xi}}_r &= \mathbf{f}_r(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots \boldsymbol{\xi}_r) + \mathbf{g}_r(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_r)\mathbf{u}_r,
\end{aligned}$$

$$\boldsymbol{\xi}_i \in \mathbb{R}^{p_i} \quad \mathbf{z}_i = (\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_i) \in \mathbb{R}^{q_i} \quad i = 0, \dots, r$$

Top-Level Safety Design

$$\mathcal{C}_0 = \{\boldsymbol{\xi}_0 \in \mathbb{R}^{p_0} \mid h_0(\boldsymbol{\xi}_0) \geq 0\}$$

$$\begin{aligned}
\frac{\partial h_0}{\partial \boldsymbol{\xi}_0}(\mathbf{z}_0) \big(\mathbf{f}_0(\mathbf{z}_0) &+ \mathbf{g}_{0,\boldsymbol{\xi}}(\mathbf{z}_0)\mathbf{k}_{0,\boldsymbol{\xi}}(\mathbf{z}_0) \\
&+ \mathbf{g}_{0,\mathbf{u}}(\mathbf{z}_0)\mathbf{k}_{0,\mathbf{u}}(\mathbf{z}_0)\big) \geq -\alpha_0(h_0(\mathbf{z}_0))
\end{aligned}$$

Multi-Step Barrier Backstepping

Caltech

Multi-Cascade System

$$\dot{\boldsymbol{\xi}}_0 = \mathbf{f}_0(\boldsymbol{\xi}_0) + \mathbf{g}_{0,\boldsymbol{\xi}}(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 + \mathbf{g}_{0,\mathbf{u}}(\boldsymbol{\xi}_0)\mathbf{u}_0,$$
$$\dot{\boldsymbol{\xi}}_1 = \mathbf{f}_1(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1) + \mathbf{g}_{1,\boldsymbol{\xi}}(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1)\boldsymbol{\xi}_2 + \mathbf{g}_{1,\mathbf{u}}(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1)\mathbf{u}_1,$$
$$\vdots$$
$$\dot{\boldsymbol{\xi}}_r = \mathbf{f}_r(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\dots\boldsymbol{\xi}_r) + \mathbf{g}_r(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_r)\mathbf{u}_r,$$

$$\boldsymbol{\xi}_i \in \mathbb{R}^{p_i} \quad \mathbf{z}_i = (\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_i) \in \mathbb{R}^{q_i} \quad i = 0,\dots,r$$

Composite Barrier

$$h(\mathbf{z}_r) = h_0(\mathbf{z}_0) - \sum_{i=1}^{r} \frac{1}{2\mu_i}\|\boldsymbol{\xi}_i - \mathbf{k}_{i-1,\boldsymbol{\xi}}(\mathbf{z}_{i-1})\|_2^2$$
$$\mu_i \in \mathbb{R}_{>0} \quad i = 1,\dots,r$$
$$\mathbf{k}_{i-1,\boldsymbol{\xi}}(\mathbf{z}_{i-1}) : \mathbb{R}^{q_{i-1}} \to \mathbb{R}^{p_i} \text{ defined recursively}$$
$$i = 2,\dots,r-1$$

Top-Level Safety Design

$$\mathcal{C}_0 = \{\boldsymbol{\xi}_0 \in \mathbb{R}^{p_0} \mid h_0(\boldsymbol{\xi}_0) \geq 0\}$$
$$\frac{\partial h_0}{\partial \boldsymbol{\xi}_0}(\mathbf{z}_0)\left(\mathbf{f}_0(\mathbf{z}_0) + \mathbf{g}_{0,\boldsymbol{\xi}}(\mathbf{z}_0)\mathbf{k}_{0,\boldsymbol{\xi}}(\mathbf{z}_0) + \mathbf{g}_{0,\mathbf{u}}(\mathbf{z}_0)\mathbf{k}_{0,\mathbf{u}}(\mathbf{z}_0)\right) \geq -\alpha_0(h_0(\mathbf{z}_0))$$

Multi-Step Barrier Backstepping

Caltech

Multi-Cascade System

$$\dot{\boldsymbol{\xi}}_0 = \mathbf{f}_0(\boldsymbol{\xi}_0) + \mathbf{g}_{0,\boldsymbol{\xi}}(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 + \mathbf{g}_{0,\mathbf{u}}(\boldsymbol{\xi}_0)\mathbf{u}_0,$$
$$\dot{\boldsymbol{\xi}}_1 = \mathbf{f}_1(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1) + \mathbf{g}_{1,\boldsymbol{\xi}}(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1)\boldsymbol{\xi}_2 + \mathbf{g}_{1,\mathbf{u}}(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1)\mathbf{u}_1,$$
$$\vdots$$
$$\dot{\boldsymbol{\xi}}_r = \mathbf{f}_r(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\dots\boldsymbol{\xi}_r) + \mathbf{g}_r(\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_r)\mathbf{u}_r,$$

$$\boldsymbol{\xi}_i \in \mathbb{R}^{p_i} \quad \mathbf{z}_i = (\boldsymbol{\xi}_0,\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_i) \in \mathbb{R}^{q_i} \quad i = 0,\dots,r$$

Composite Barrier

$$h(\mathbf{z}_r) = h_0(\mathbf{z}_0) - \sum_{i=1}^{r} \frac{1}{2\mu_i}\|\boldsymbol{\xi}_i - \mathbf{k}_{i-1,\boldsymbol{\xi}}(\mathbf{z}_{i-1})\|_2^2$$
$$\mu_i \in \mathbb{R}_{>0} \quad i = 1,\dots,r$$
$$\mathbf{k}_{i-1,\boldsymbol{\xi}}(\mathbf{z}_{i-1}) : \mathbb{R}^{q_{i-1}} \to \mathbb{R}^{p_i} \text{ defined recursively}$$
$$i = 2,\dots,r-1$$

Top-Level Safety Design

$$\mathcal{C}_0 = \{\boldsymbol{\xi}_0 \in \mathbb{R}^{p_0} \mid h_0(\boldsymbol{\xi}_0) \geq 0\}$$
$$\frac{\partial h_0}{\partial \boldsymbol{\xi}_0}(\mathbf{z}_0)\left(\mathbf{f}_0(\mathbf{z}_0) + \mathbf{g}_{0,\boldsymbol{\xi}}(\mathbf{z}_0)\mathbf{k}_{0,\boldsymbol{\xi}}(\mathbf{z}_0) + \mathbf{g}_{0,\mathbf{u}}(\mathbf{z}_0)\mathbf{k}_{0,\mathbf{u}}(\mathbf{z}_0)\right) \geq -\alpha_0(h_0(\mathbf{z}_0))$$

Composite Safe Set

$$\mathcal{C} = \{\mathbf{z}_r \in \mathbb{R}^{q_r} \mid h(\mathbf{z}_r) \geq 0\}$$
$$\mathcal{C} \subseteq \mathcal{C}_0 \times \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_r}$$

Joint Lyapunov-Barrier Backstepping

Caltech

Lyapunov & Barrier Top Level Design

$$\dot{V}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \leq -\gamma(V_0(\mathbf{x}_0))$$
$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$

Joint Lyapunov-Barrier Backstepping

Caltech

Lyapunov & Barrier Top Level Design

$$\dot{V}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \leq -\gamma(V_0(\mathbf{x}_0))$$
$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$

⇓

Composite Lyapunov & Barrier

$$V(\mathbf{x}, \boldsymbol{\xi}) = V_0(\mathbf{x}) + \frac{1}{2\mu_V}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$
$$h(\mathbf{x}, \boldsymbol{\xi}) = h_0(\mathbf{x}) - \frac{1}{2\mu_h}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$

Joint Lyapunov-Barrier Backstepping

Caltech

Lyapunov & Barrier Top Level Design

$$\dot{V}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \leq -\gamma(V_0(\mathbf{x}_0))$$
$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$

Composite Lyapunov & Barrier

$$V(\mathbf{x}, \boldsymbol{\xi}) = V_0(\mathbf{x}) + \frac{1}{2\mu_V}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$
$$h(\mathbf{x}, \boldsymbol{\xi}) = h_0(\mathbf{x}) - \frac{1}{2\mu_h}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$

Time Derivatives

$$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) = b_{V,1}(\mathbf{x}, \boldsymbol{\xi}) + \frac{1}{\mu_V}\mathbf{a}_1(\mathbf{x}, \boldsymbol{\xi})^\top \mathbf{u}$$
$$\dot{h}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) = b_{h,1}(\mathbf{x}, \boldsymbol{\xi}) - \frac{1}{\mu_h}\mathbf{a}_1(\mathbf{x}, \boldsymbol{\xi})^\top \mathbf{u}$$

Joint Lyapunov-Barrier Backstepping

Caltech

Lyapunov & Barrier Top Level Design

$$\dot{V}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \leq -\gamma(V_0(\mathbf{x}_0))$$
$$\dot{h}_0(\mathbf{x}, \mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$

Composite Lyapunov & Barrier

$$V(\mathbf{x}, \boldsymbol{\xi}) = V_0(\mathbf{x}) + \frac{1}{2\mu_V}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$
$$h(\mathbf{x}, \boldsymbol{\xi}) = h_0(\mathbf{x}) - \frac{1}{2\mu_h}(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))$$

Time Derivatives

$$\dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) = b_{V,1}(\mathbf{x}, \boldsymbol{\xi}) + \frac{1}{\mu_V}\mathbf{a}_1(\mathbf{x}, \boldsymbol{\xi})^\top \mathbf{u}$$
$$\dot{h}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) = b_{h,1}(\mathbf{x}, \boldsymbol{\xi}) - \frac{1}{\mu_h}\mathbf{a}_1(\mathbf{x}, \boldsymbol{\xi})^\top \mathbf{u}$$

CLF + CBF Condition

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \dot{V}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) = b_{V,1}(\mathbf{x}, \boldsymbol{\xi}) + \frac{1}{\mu_V}\mathbf{a}_1(\mathbf{x}, \boldsymbol{\xi})^\top \mathbf{u} \leq -\gamma_3(\|\mathbf{x}\|) - \gamma_3'(\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x}))\|)$$

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \dot{h}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}) = -b_{h,1}(\mathbf{x}, \boldsymbol{\xi}) + \frac{1}{\mu_h}\mathbf{a}_1(\mathbf{x}, \boldsymbol{\xi})^\top \mathbf{u} \leq \alpha_0(h_0(\mathbf{x})) - \frac{\lambda}{2\mu_h}\|\boldsymbol{\xi} - \mathbf{k}_0(\mathbf{x})\|_2^2$$

Joint Lyapunov-Barrier Backstepping

Caltech

Lyapunov & Barrier Top Level Design

$$\dot{V}_0(\mathbf{x},\mathbf{k}_0(\mathbf{x})) \leq -\gamma(V_0(\mathbf{x}_0))$$
$$\dot{h}_0(\mathbf{x},\mathbf{k}_0(\mathbf{x})) \geq -\alpha_0(h(\mathbf{x}))$$

↓

Composite Lyapunov & Barrier

$$V(\mathbf{x},\boldsymbol{\xi}) = V_0(\mathbf{x}) + \frac{1}{2\mu_V}(\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x}))$$
$$h(\mathbf{x},\boldsymbol{\xi}) = h_0(\mathbf{x}) - \frac{1}{2\mu_h}(\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x}))^\top(\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x}))$$

↓

Time Derivatives

$$\dot{V}(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) = b_{V,1}(\mathbf{x},\boldsymbol{\xi}) + \frac{1}{\mu_V}\mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top\mathbf{u}$$
$$\dot{h}(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) = b_{h,1}(\mathbf{x},\boldsymbol{\xi}) - \frac{1}{\mu_h}\mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top\mathbf{u}$$

→

CLF + CBF Condition

$$\inf_{\mathbf{u}\in\mathbb{R}^m} \dot{V}(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) = b_{V,1}(\mathbf{x},\boldsymbol{\xi}) + \frac{1}{\mu_V}\mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top\mathbf{u}$$
$$\leq -\gamma_3(\|\mathbf{x}\|) - \gamma_3'(\|\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x})\|)$$
$$\inf_{\mathbf{u}\in\mathbb{R}^m} \dot{h}(\mathbf{x},\boldsymbol{\xi},\mathbf{u}) = -b_{h,1}(\mathbf{x},\boldsymbol{\xi}) + \frac{1}{\mu_h}\mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top\mathbf{u}$$
$$\leq \alpha_0(h_0(\mathbf{x})) - \frac{\lambda}{2\mu_h}\|\boldsymbol{\xi}-\mathbf{k}_0(\mathbf{x})\|_2^2$$

↓

Rewrite Conditions

$$\inf_{\mathbf{u}\in\mathbb{R}^m} \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top\mathbf{u} \leq c_{V,1}(\mathbf{x},\boldsymbol{\xi})$$
$$\inf_{\mathbf{u}\in\mathbb{R}^m} \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top\mathbf{u} \leq c_{h,1}(\mathbf{x},\boldsymbol{\xi})$$

Joint Lyapunov-Barrier Backstepping

Caltech

Rewrite Conditions

$$\inf_{\mathbf{u}\in\mathbb{R}^m} \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top \mathbf{u} \leq c_{V,1}(\mathbf{x},\boldsymbol{\xi})$$

$$\inf_{\mathbf{u}\in\mathbb{R}^m} \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top \mathbf{u} \leq c_{h,1}(\mathbf{x},\boldsymbol{\xi})$$

Stability and safety conditions are jointly satisfiable!

Stabilizing + Safe Backstepping Controller

$$\mathbf{k}(\mathbf{x},\boldsymbol{\xi}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2 \quad \text{s.t. } \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top \mathbf{u} \leq \min\{c_{V,1}(\mathbf{x},\boldsymbol{\xi}), c_{h,1}(\mathbf{x},\boldsymbol{\xi})\}$$

Joint Lyapunov-Barrier Backstepping

Caltech

Rewrite Conditions

$$\inf_{\mathbf{u}\in\mathbb{R}^m} \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top \mathbf{u} \leq c_{V,1}(\mathbf{x},\boldsymbol{\xi})$$

$$\inf_{\mathbf{u}\in\mathbb{R}^m} \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top \mathbf{u} \leq c_{h,1}(\mathbf{x},\boldsymbol{\xi})$$

Stability and safety conditions are jointly satisfiable!

Tracking the stabilizing and safe top-level controller faster benefits both stability and safety.

Stabilizing + Safe Backstepping Controller

$$\mathbf{k}(\mathbf{x},\boldsymbol{\xi}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2 \quad \text{s.t. } \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top \mathbf{u} \leq \min\{c_{V,1}(\mathbf{x},\boldsymbol{\xi}), c_{h,1}(\mathbf{x},\boldsymbol{\xi})\}$$

Joint Lyapunov-Barrier Backstepping

Caltech

Rewrite Conditions

$$\inf_{\mathbf{u}\in\mathbb{R}^m} \mathbf{a}_1(\mathbf{x},\boldsymbol{\xi})^\top \mathbf{u} \leq c_{V,1}(\mathbf{x},\boldsymbol{\xi})$$

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Tracking the stabilizing and safe top-level controller faster benefits both stability and safety.

How do we design a smooth top-level controller meeting both constraints?

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Tracking the stabilizing and safe top-level controller faster benefits both stability and safety.

Optimization-based controllers generally are not smooth.

How do we design a smooth top-level controller meeting both constraints?

Smooth Control Synthesis

Caltech

Top-Level System Joint CLF + CBF

For all $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{v} \in \mathbb{R}^p$ s.t.

$$\frac{\partial V_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{v})) \leq -\gamma_3(\|\mathbf{x}\|)$$

$$\frac{\partial h_0}{\partial \mathbf{x}}(\mathbf{x})(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{v})) \geq -\alpha_0(h_0(\mathbf{x}))$$

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↓

Rewrite Constraints

$$\mathbf{a}_{V,0}(\mathbf{x})^\top \mathbf{v} + b_{V,0}(\mathbf{x}) \leq 0$$

$$\mathbf{a}_{h,0}(\mathbf{x})^\top \mathbf{v} + b_{h,0}(\mathbf{x}) \leq 0$$

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$$\mathbf{a}_{h,0}(\mathbf{x})^\top \mathbf{v} + b_{h,0}(\mathbf{x}) \leq 0$$

Feasible Input Sets

$$\mathcal{U}_V(\mathbf{x}) = \{\mathbf{v} \in \mathbb{R}^p \mid \mathbf{a}_{V,0}(\mathbf{x})^\top \mathbf{v} + b_{V,0}(\mathbf{x}) \leq 0\}$$

$$\mathcal{U}_h(\mathbf{x}) = \{\mathbf{v} \in \mathbb{R}^p \mid \mathbf{a}_{h,0}(\mathbf{x})^\top \mathbf{v} + b_{h,0}(\mathbf{x}) \leq 0\}$$

$$\mathcal{U}_h(\mathbf{x}) \cap \mathcal{U}_V(\mathbf{x}) \neq \emptyset$$

Smooth Control Synthesis

Partition of Unity Approach[8]

Caltech

Top-Level System Joint CLF + CBF

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Feasible Input Sets

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$$\mathcal{U}_h(\mathbf{x}) = \{\mathbf{v} \in \mathbb{R}^p \mid \mathbf{a}_{h,0}(\mathbf{x})^\top \mathbf{v} + b_{h,0}(\mathbf{x}) \leq 0\}$$

$$\mathcal{U}_h(\mathbf{x}) \cap \mathcal{U}_V(\mathbf{x}) \neq \emptyset$$

[8] P. Ong, J. Cortés, "Universal formula for smooth safe stabilization", 2019.

Smooth Control Synthesis

Partition of Unity Approach[8]

Caltech

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Feasible Input Sets

$$\mathcal{U}_V(\mathbf{x}) = \{\mathbf{v} \in \mathbb{R}^p \mid \mathbf{a}_{V,0}(\mathbf{x})^\top \mathbf{v} + b_{V,0}(\mathbf{x}) \leq 0\}$$

$$\mathcal{U}_h(\mathbf{x}) = \{\mathbf{v} \in \mathbb{R}^p \mid \mathbf{a}_{h,0}(\mathbf{x})^\top \mathbf{v} + b_{h,0}(\mathbf{x}) \leq 0\}$$

$$\mathcal{U}_h(\mathbf{x}) \cap \mathcal{U}_V(\mathbf{x}) \neq \emptyset$$

Gaussian Weighted Centroid[9,10]

$$\boldsymbol{\mu}(\mathbf{x};\mathcal{U}) = \frac{\int_\mathcal{U} \mathbf{v}\phi(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}}{\int_\mathcal{U} \phi(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}} \qquad \phi(\mathbf{x},\mathbf{v}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\|\mathbf{v}\|_2^2}{2\sigma(\mathbf{x})}}$$

$$\boldsymbol{\mu} : \mathbb{R}^n \to \mathbb{R}^p \qquad \mathcal{U} \subseteq \mathbb{R}^p \qquad \phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$$

$$\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \text{ smooth}$$

[8] P. Ong, J. Cortés, "Universal formula for smooth safe stabilization", 2019.
[9] G. M. Tallis, "The moment generating function of the truncated multi-normal distribution", 1961.
[10] G. M. Tallis, "Plane truncation in normal populations", 1965.

Smooth Control Synthesis

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Controller Design[8]

$$\mathbf{k}_0(\mathbf{x}) = \zeta(\rho(\mathbf{x}))(\boldsymbol{\mu}(\mathbf{x};\mathcal{U}_V) + \boldsymbol{\mu}(\mathbf{x};\mathcal{U}_h)) \\ +(1-\zeta(\rho(\mathbf{x})))\boldsymbol{\mu}(\mathbf{x};\mathcal{U}_V \cap \mathcal{U}_h)$$

$\zeta : \mathbb{R} \to [0,1]$ smooth partition of unity

$$\rho(\mathbf{x}) = \frac{\mathbf{a}_{V,0}(\mathbf{x})^\top \mathbf{a}_{h,0}(\mathbf{x})}{\|\mathbf{a}_{V,0}(\mathbf{x})\|_2 \|\mathbf{a}_{h,0}(\mathbf{x})\|_2}$$

[8] P. Ong, J. Cortés, "Universal formula for smooth safe stabilization", 2019.

Smooth Control Synthesis

Caltech

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$\zeta : \mathbb{R} \to [0,1]$ smooth partition of unity

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Stability + Safety + Smoothness[8]

$$\mathbf{k}_0(\mathbf{x}) \in \mathcal{U}_V(\mathbf{x}) \cap \mathcal{U}_h(\mathbf{x})$$

$\mathbf{k}_0$ is smooth*

*Special considerations for origin addressed in paper

[8] P. Ong, J. Cortés, "Universal formula for smooth safe stabilization", 2019.

Simulation Results

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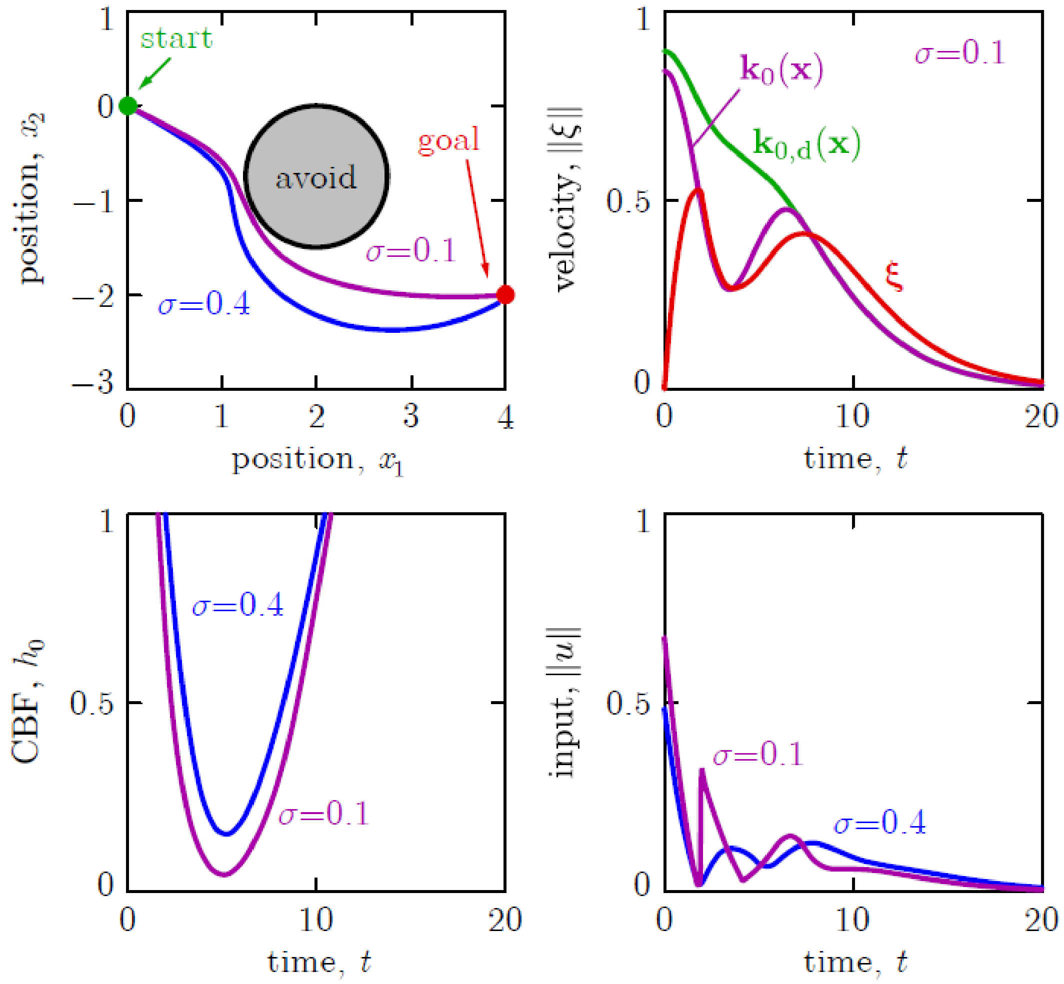
Double Integrator

$$\dot{\mathbf{x}} = \boldsymbol{\xi}$$
$$\dot{\boldsymbol{\xi}} = \mathbf{u}$$
$$h_0(\mathbf{x}) = \frac{1}{2}\left(\|\mathbf{x} - \mathbf{x}_\mathrm{O}\|_2^2 - R_\mathrm{O}^2\right)$$

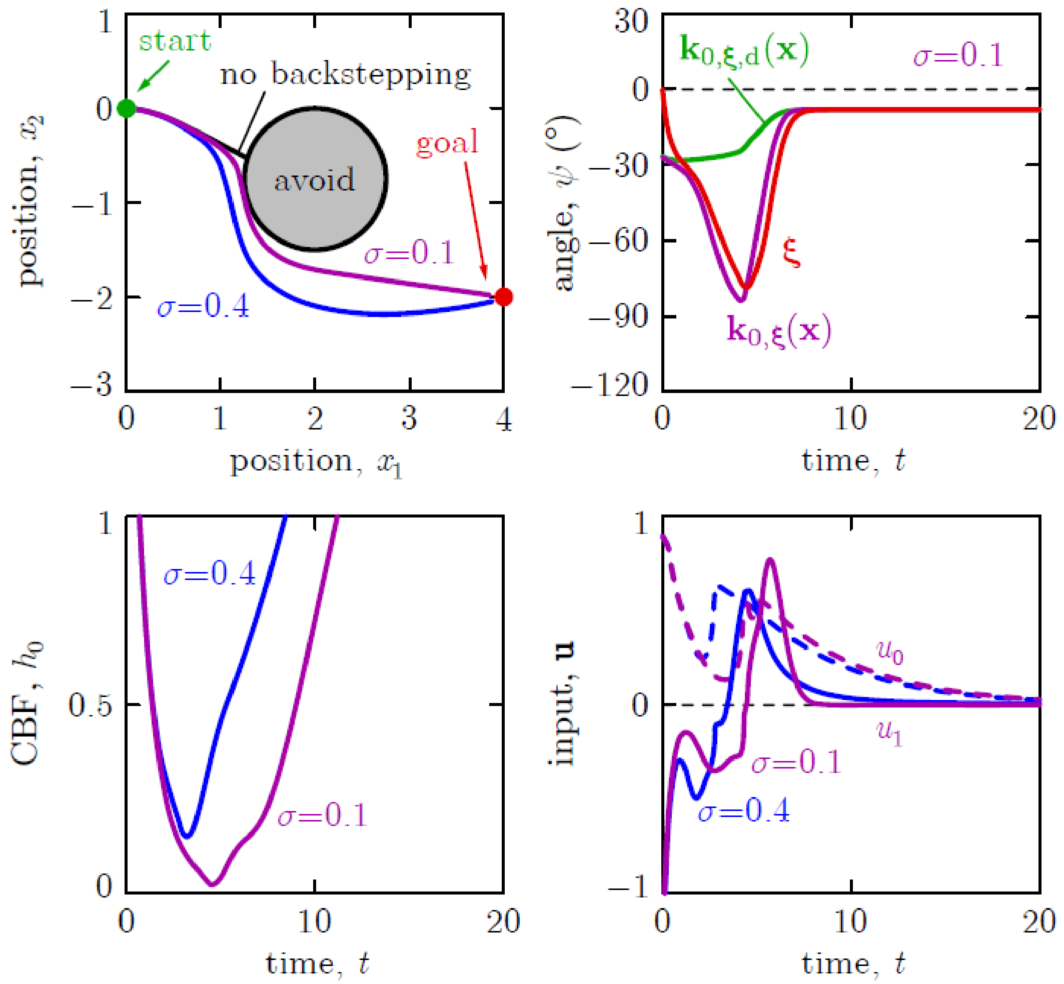


Simulation Results

Caltech

Unicycle

$$\dot{x} = v\cos(\psi)$$
$$\dot{y} = v\sin(\psi)$$
$$\dot{\psi} = \omega$$
$$h_0(\mathbf{x}) = \frac{1}{2}\left(\|\mathbf{x} - \mathbf{x}_O\|_2^2 - R_O^2\right)$$



Conclusions

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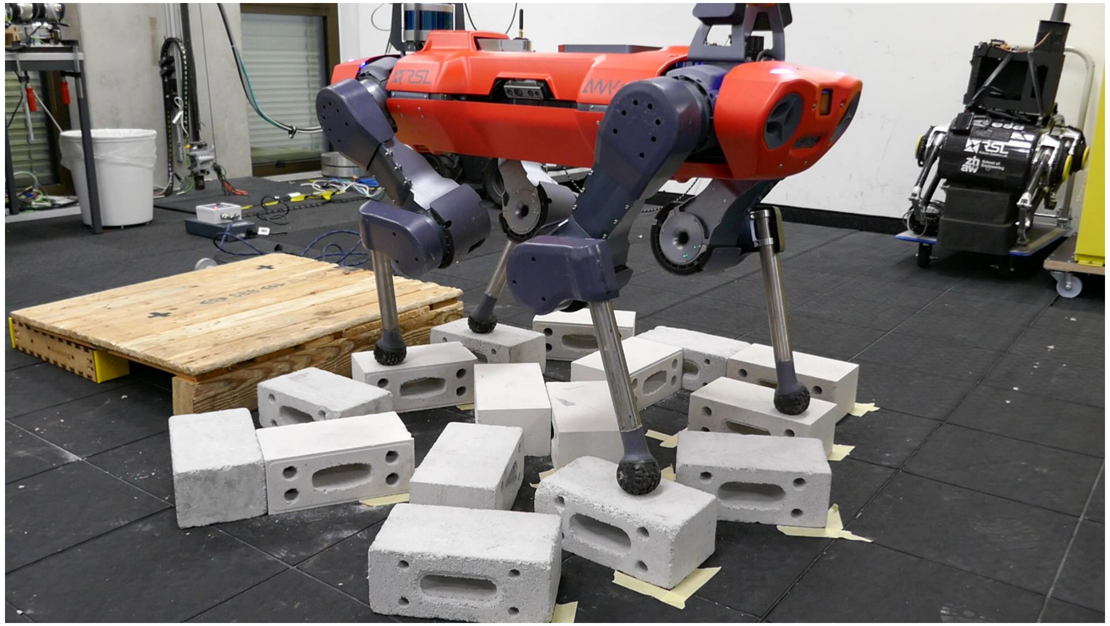
- Framework for achieving safety of higher-order systems by unifying classical **Lyapunov backstepping** with **Control Barrier Functions**

- Constructive tool for **synthesizing** Control Barrier Functions for higher-order systems

- Design of **stable and safe** nonlinear controllers through joint Lyapunov and Barrier backstepping



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Thank You!

Safe Backstepping with Control Barrier Functions

Andrew J. Taylor Pio Ong Tamas G. Molnár
Aaron D. Ames