Towards Robust Data-Driven Control Synthesis for Nonlinear Systems with Actuation Uncertainty

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University of California at Berkeley

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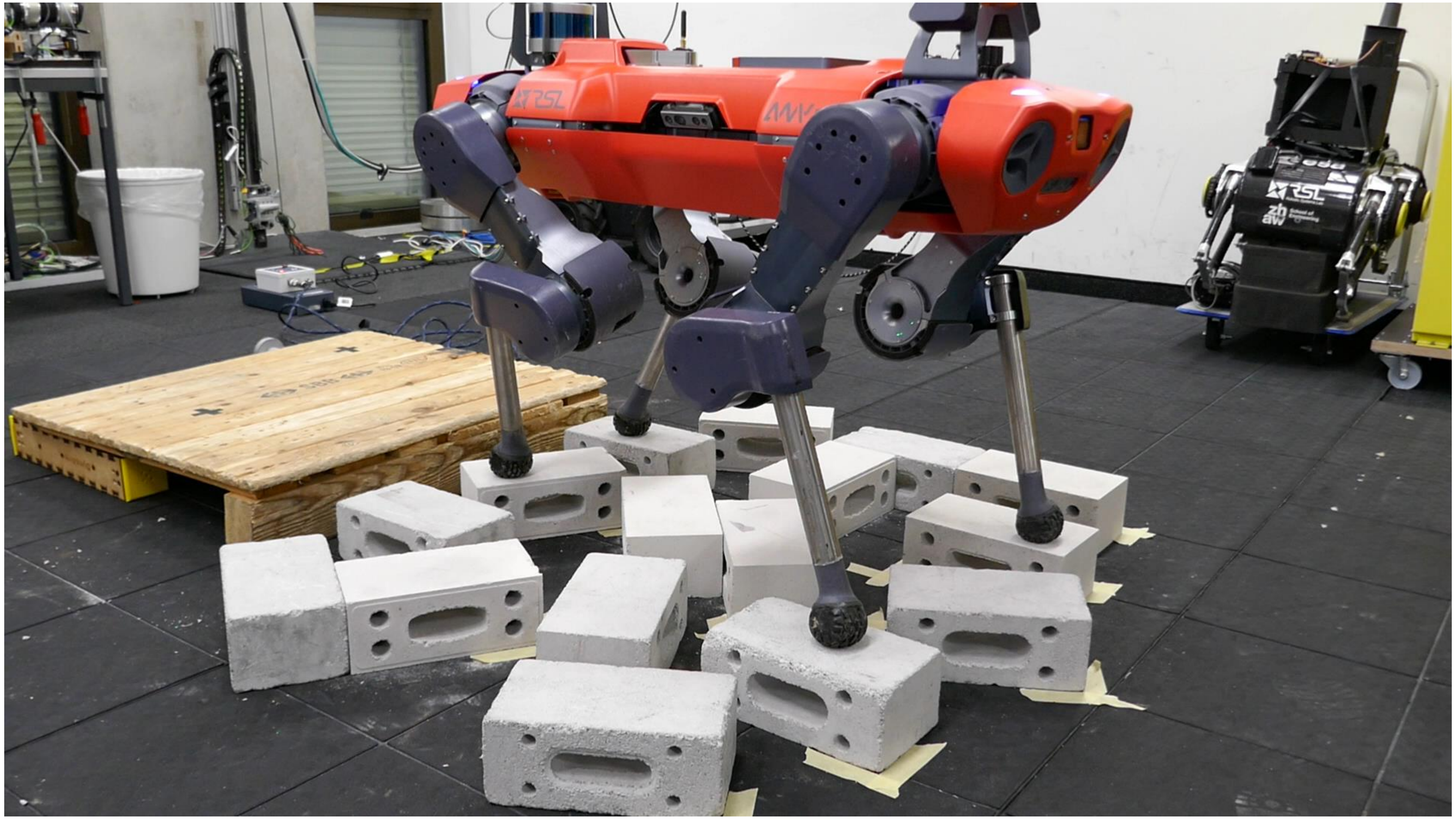
Control & Decision Conference (CDC) 2021

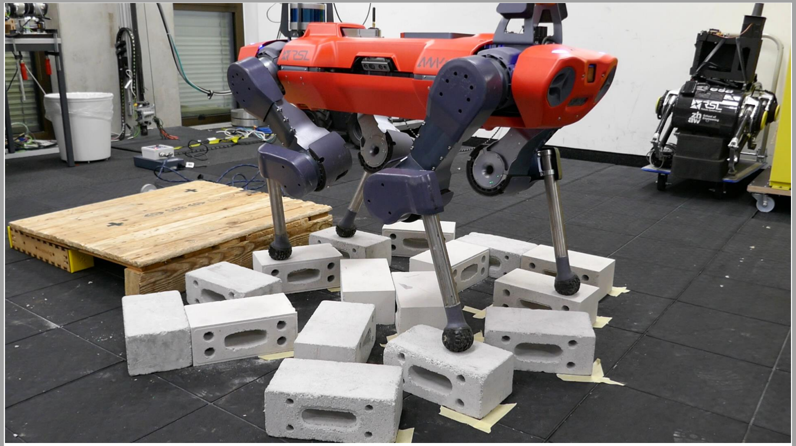
Control in the real world is hard

But: Pretty when it works…



[1] R. Grandia, **A. J. Taylor**, M. Hutter, A. D. Ames, "Multi-Layered Safety for Legged Robotics via Control Barrier Functions and Model Predictive Control", 2020.

Claim: Need to Bridge the Gap



$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2$$
$$\text{s.t.} \quad \nabla \mathtt{C}(\mathbf{x})(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}) \leq -\alpha(\mathtt{C}(\mathbf{x}))$$

Bridge the Gap

$$\mathbf{k}_{\mathrm{rob}}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2$$
$$\text{s.t.} \quad \widehat{\dot{\mathtt{C}}}(\mathbf{x},\mathbf{u}) + \nabla \mathtt{C}(\mathbf{x})(\mathbf{b} + \mathbf{A}\mathbf{u}) \leq -\alpha(\mathtt{C}(\mathbf{x}))$$
$$\text{for all } (\mathbf{A},\mathbf{b}) \in \mathcal{U}(\mathbf{x})$$

Theorems & Proofs

Experimental Realization

Contributions

- Framework for achieving robust data-driven control for stability and safety via **Control Certificate Functions (CCFs)**

- Formulation of nonlinear controller directly incorporating data through robust convex optimization.

- Analysis of controller feasibility based on dataset properties and uncertainty quantification.

System Dynamics

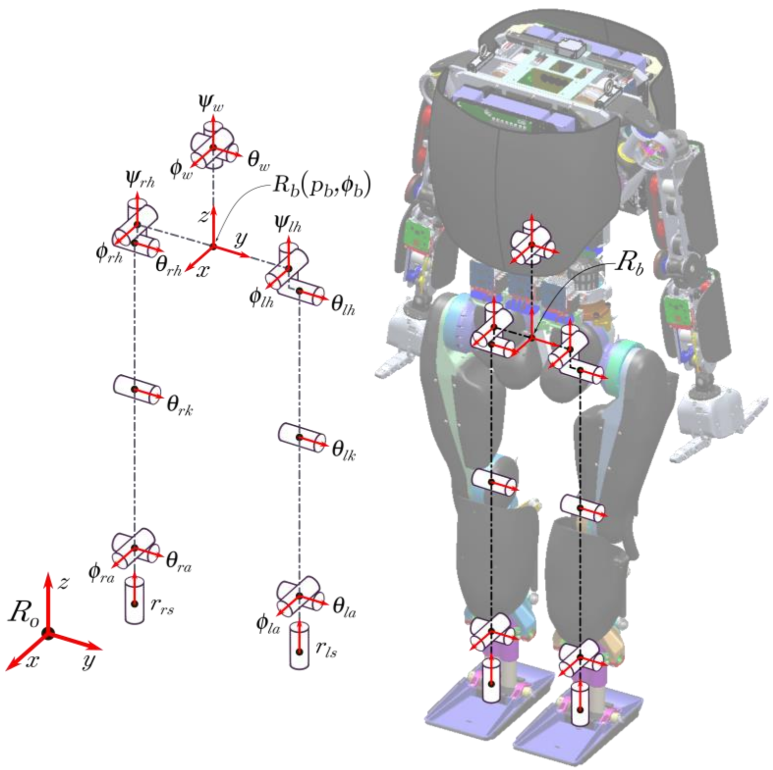
Equations of Motion

$$\dot{\widehat{\mathbf{x}}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}$$

$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$

$$\widehat{\mathbf{f}} : \mathbb{R}^n \to \mathbb{R}^n \quad \widehat{\mathbf{g}} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$$

Mathematical Model



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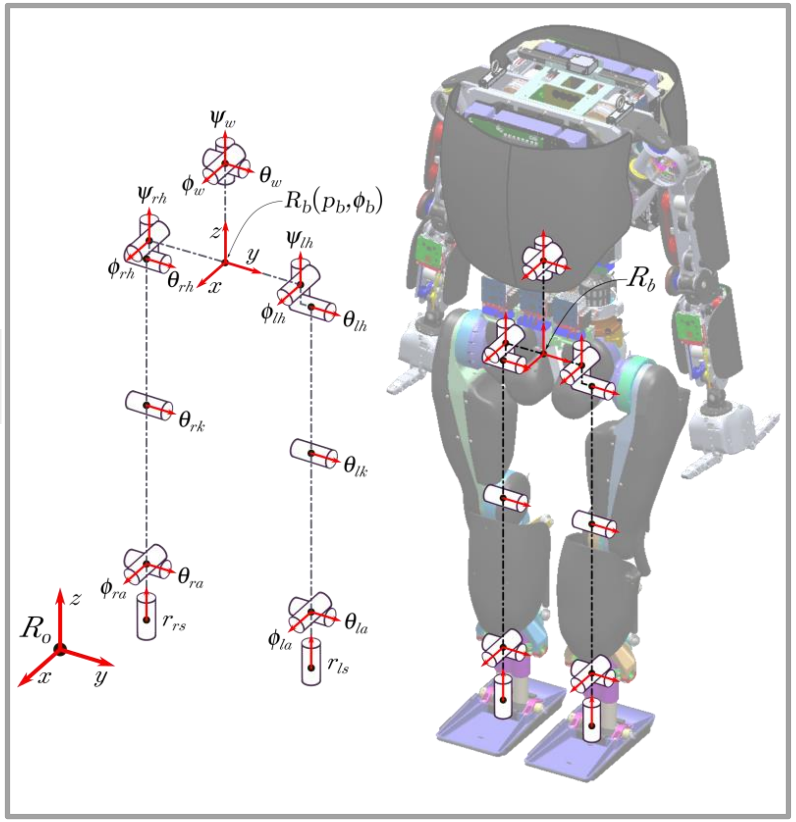
$$\widehat{\mathbf{f}} : \mathbb{R}^n \to \mathbb{R}^n \quad \widehat{\mathbf{g}} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$$

Assumptions

$\widehat{\mathbf{f}}$, $\widehat{\mathbf{g}}$ locally Lipschitz continuous

$$\widehat{\mathbf{f}}(\mathbf{0}) = \mathbf{0}$$

Mathematical Model



System Model

Control Certificate Functions (CCFs)

Certificate Function

$$\mathtt{C} : \mathbb{R}^n \to \mathbb{R}$$

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Certificate Function

$$\mathtt{C} : \mathbb{R}^n \to \mathbb{R}$$

Comparison Function

$$\alpha : \mathbb{R} \to \mathbb{R}$$

Control Certificate Functions (CCFs)

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Encode control objectives!

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Closed-Loop

$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{k}(\mathbf{x})$

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CF Condition

$$\widehat{\dot{\mathtt{C}}}(\mathbf{x}) = \nabla \mathtt{C}(\mathbf{x})(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{k}(\mathbf{x})) \leq -\alpha(\mathtt{C}(\mathbf{x}))$$

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Recent Work

[2] R. Dimitrova, R. Majumdar, "Deductive control synthesis for alternating-time logics", 2014.
[3] N. M. Boffi, et al., "Learning stability certificates from data", 2020.

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Control Lyapunov Functions (CLFs)

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Control Lyapunov Function

$$\alpha_1(\|\mathbf{x}\|) \leq \mathtt{C}(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$$

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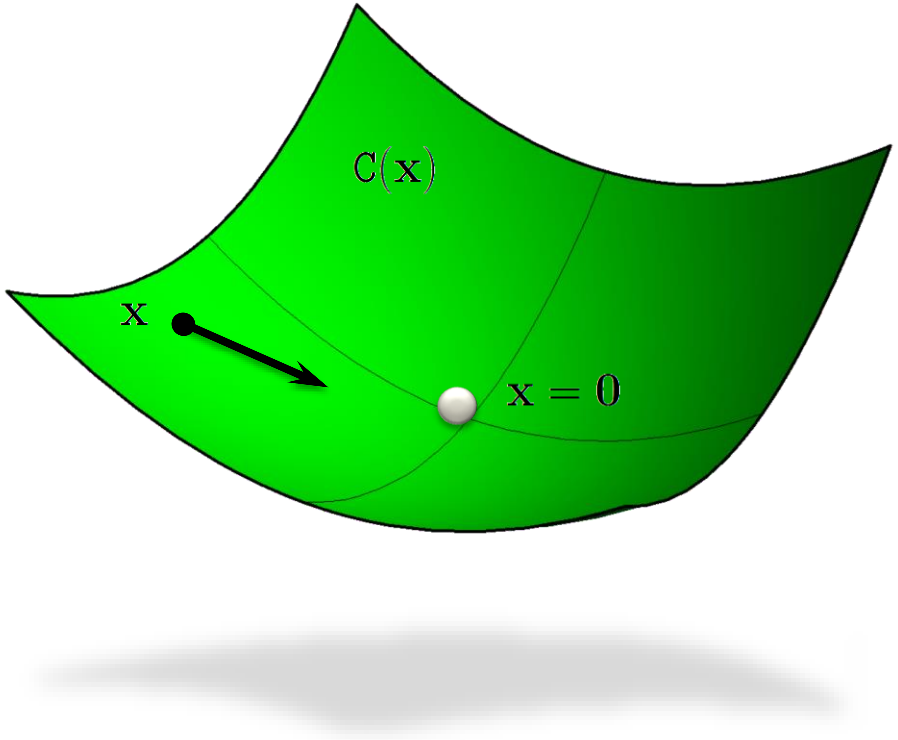
$$\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$$



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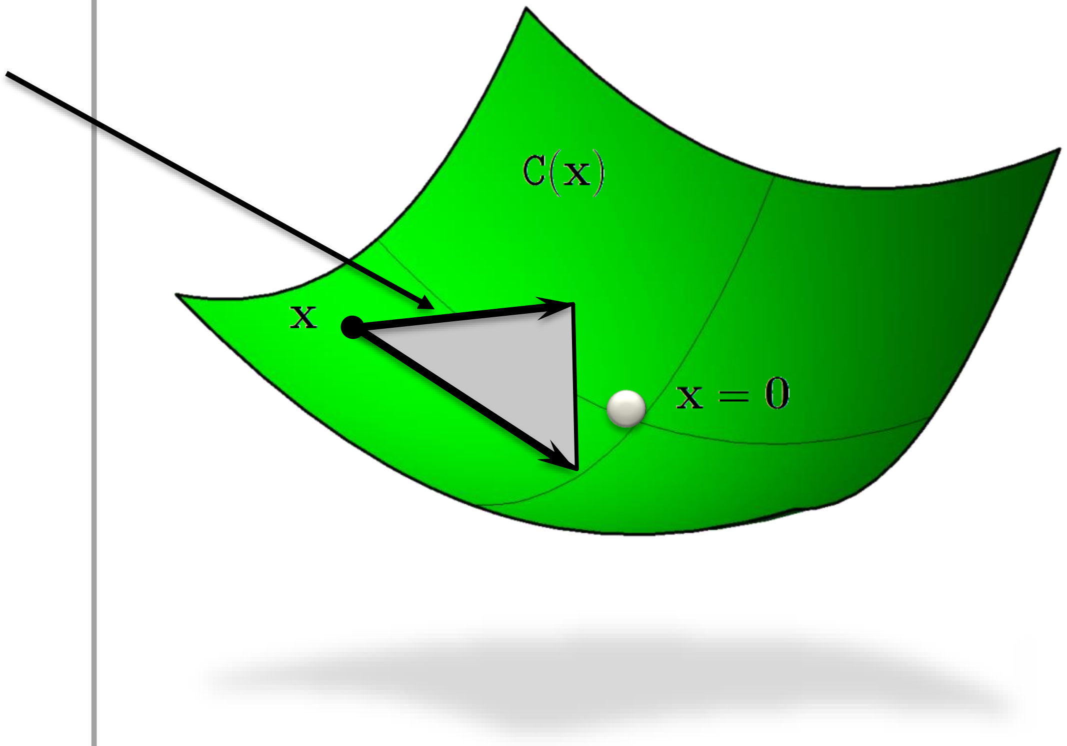

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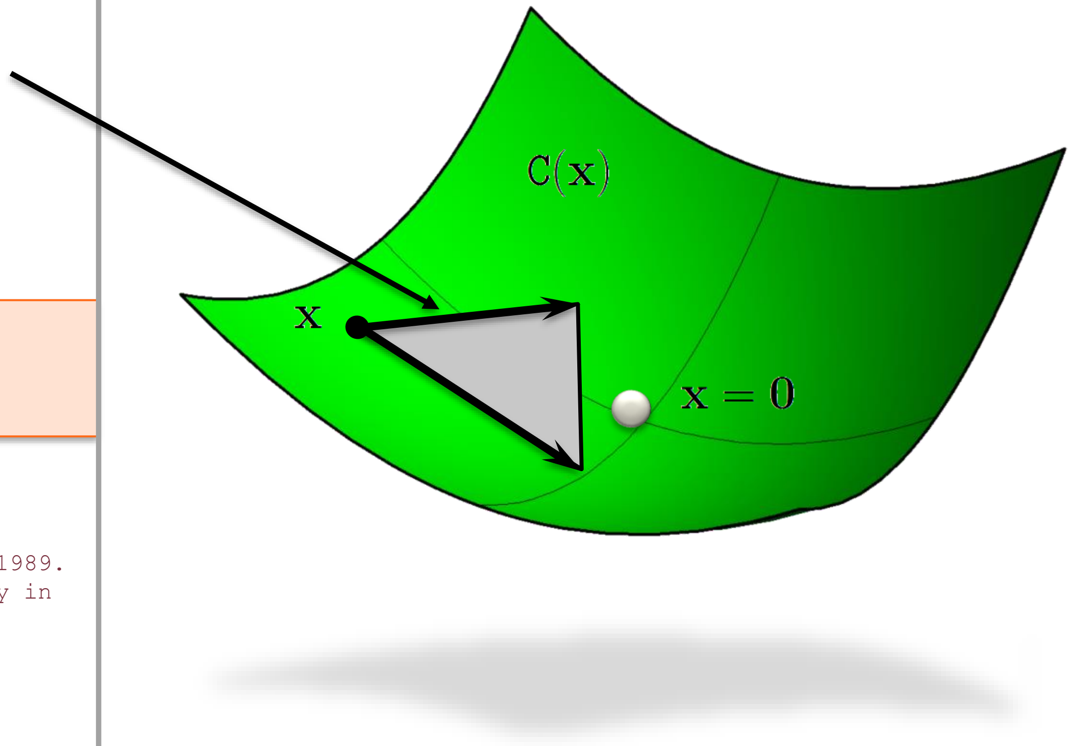
Feedback Controllers

[4] Z. Artstein, "Stabilization with relaxed controls", 1983.
[5] E. Sontag, "A universal construction of Artstein's theorem on nonlinear stabilization", 1989.
[6] R. Freeman, P. Kokotovic, "Inverse Optimality in Robust Stabilization", 1996.



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CLF Quadratic Program[7]

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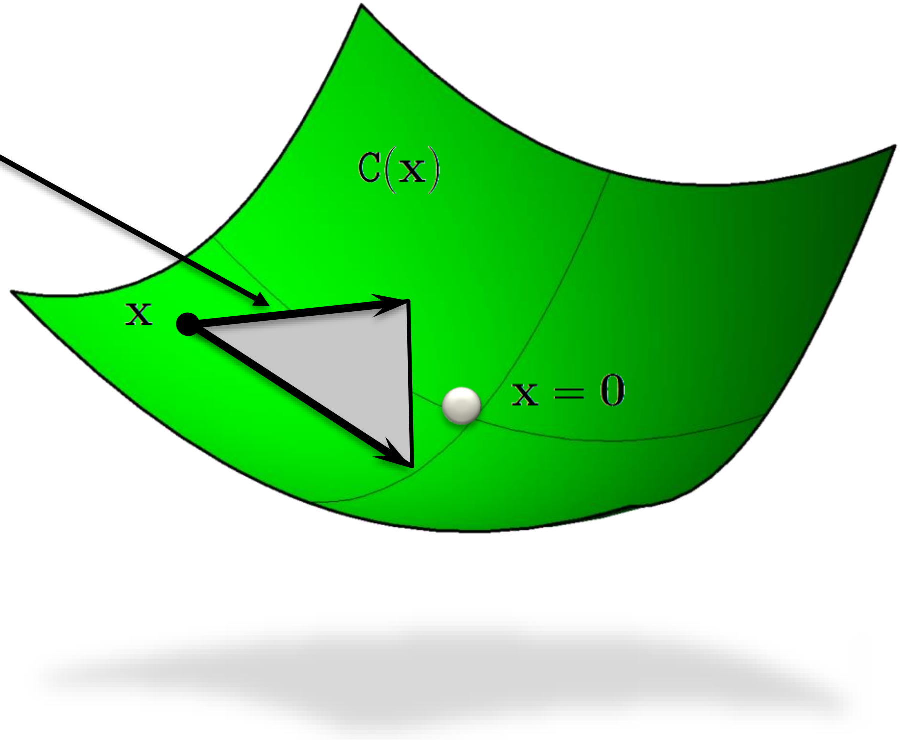

[7] A. Ames, M. Powell, "Towards the unification of locomotion and manipulation through control lyapunov functions and quadratic programs", 2013.

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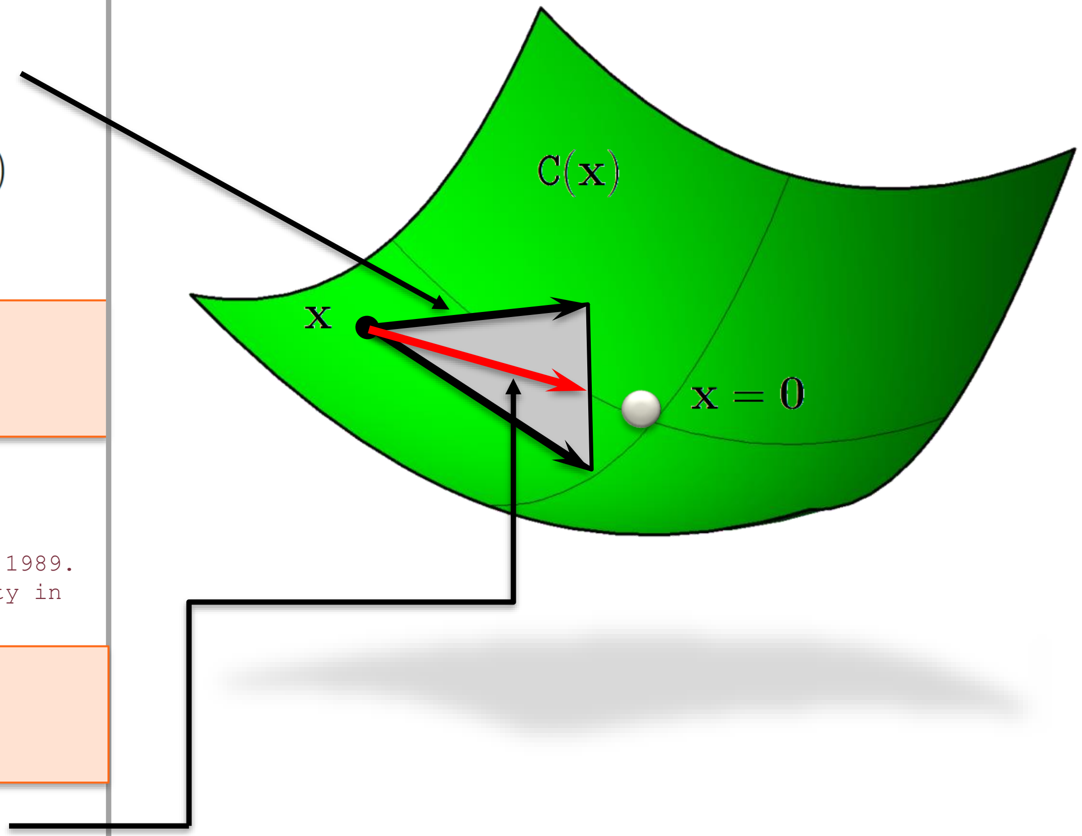

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Control Barrier Functions (CBFs)

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$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathtt{C}(\mathbf{x}) \leq 0\}$$

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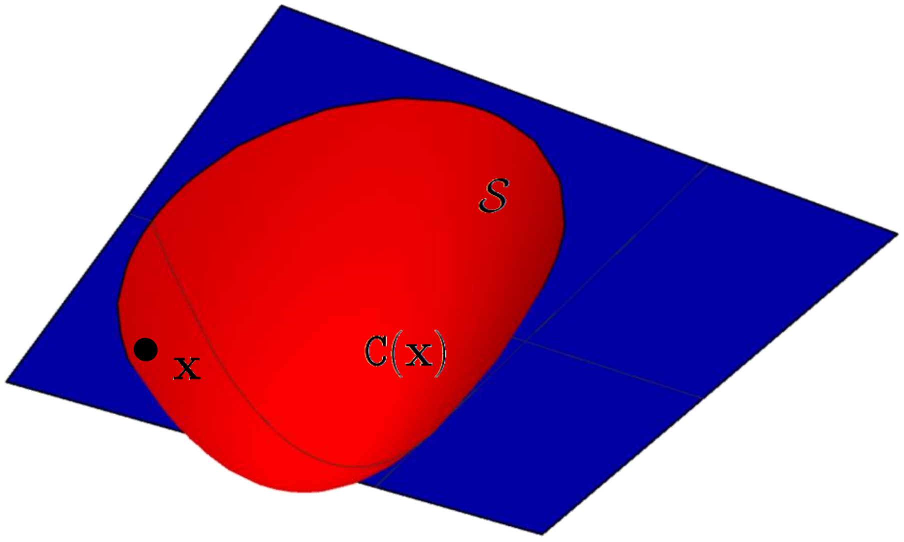

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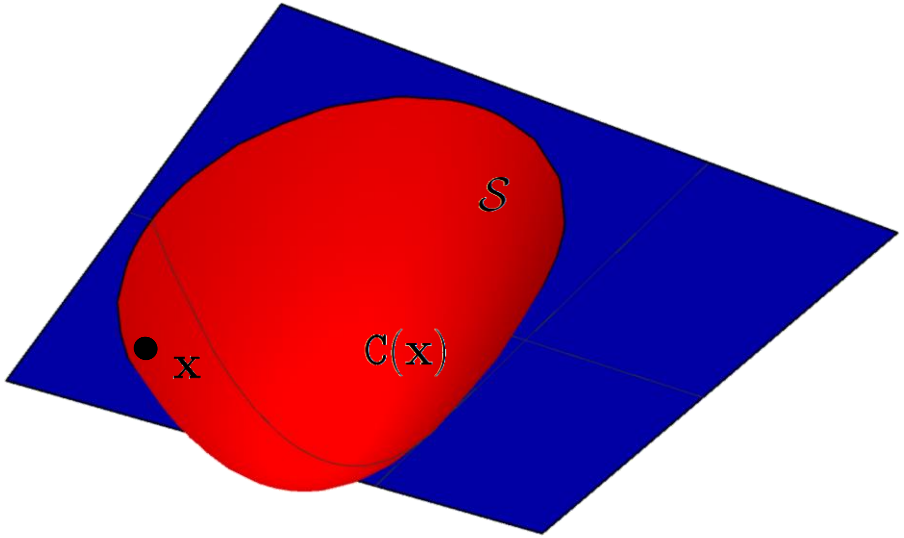
Safety = Forward Invariance



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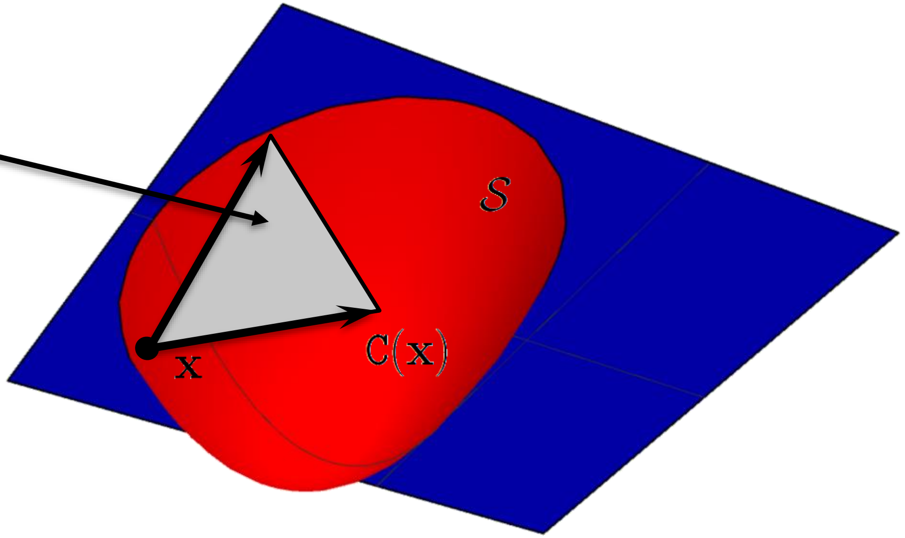

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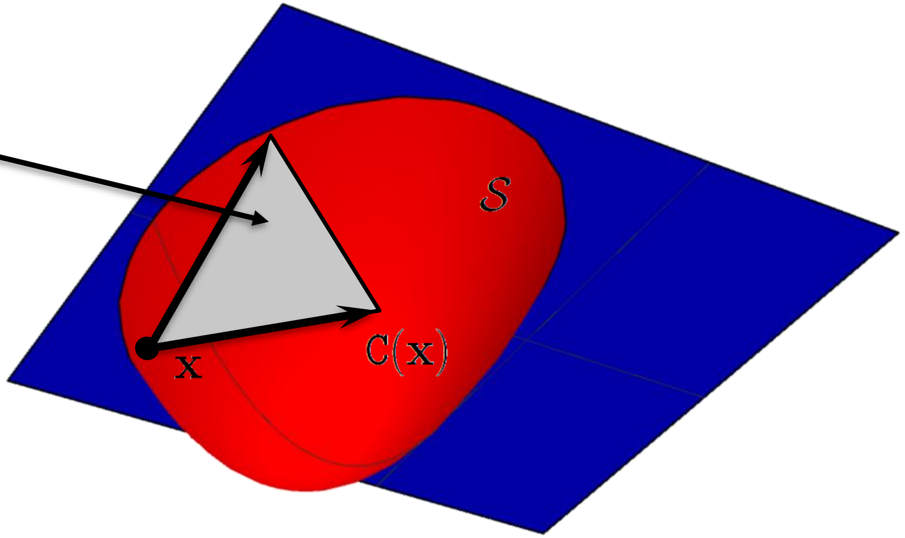

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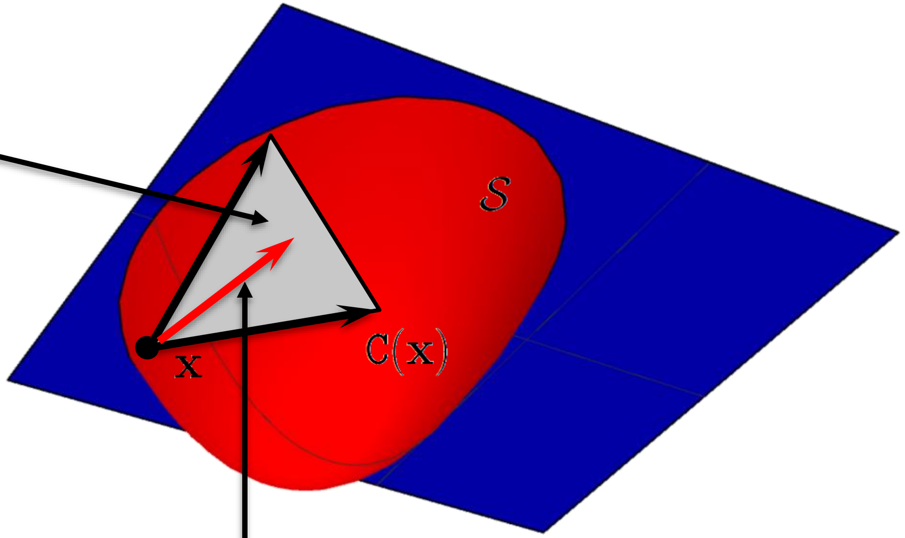

Uncertain System Dynamics

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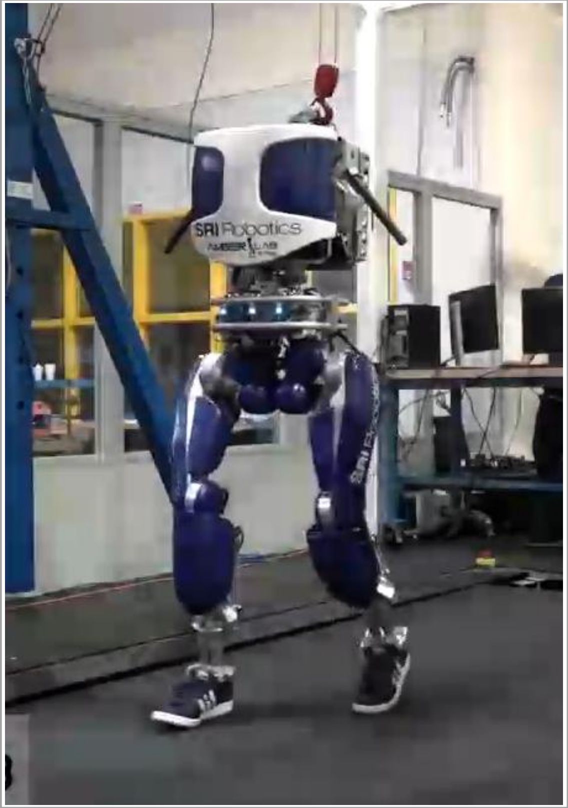
True Dynamics



Physical Robot

Uncertain System Dynamics

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Methods

- Adaptive Control [9]
- System Identification [10]
- Machine Learning [11]
- High-gain control [12]

True Dynamics

Physical Robot

[9] M. Krstic, et al., "Nonlinear Adaptive Control Design"
[10] L. Ljung, "System Identification"
[11] J. Kober, et al., "Reinforcement learning in robotics: A survey"
[12] A. Ilchmann, et al., "High-gain control without identification: a survey"

Uncertain System Dynamics

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Assumptions

$$\widetilde{\mathbf{f}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \widehat{\mathbf{f}}(\mathbf{x}) \quad \widetilde{\mathbf{g}}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) - \widehat{\mathbf{g}}(\mathbf{x})$$

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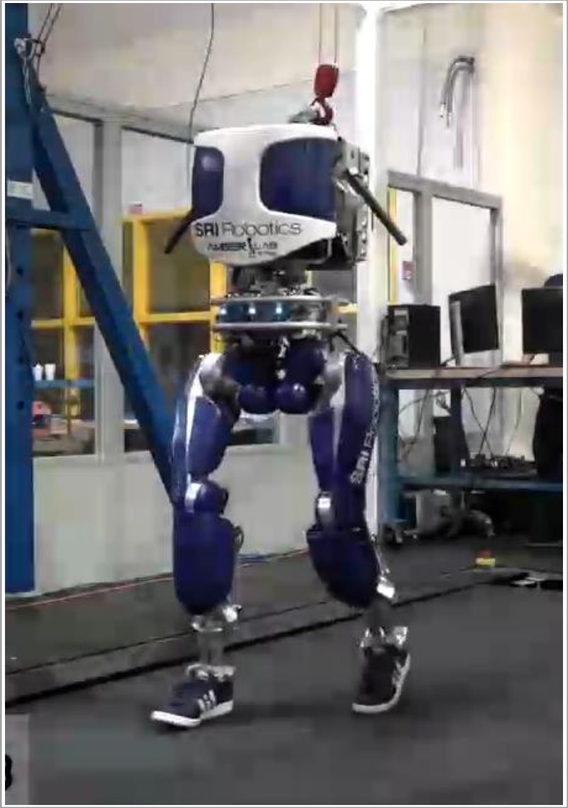
True Dynamics



Physical Robot

Uncertain System Dynamics

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Assumptions
$\widetilde{\mathbf{f}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \widehat{\mathbf{f}}(\mathbf{x}) \quad \widetilde{\mathbf{g}}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) - \widehat{\mathbf{g}}(\mathbf{x})$ $\widetilde{\mathbf{f}},\ \widetilde{\mathbf{g}}$ Lipschitz continuous, $\mathcal{L}_{\widetilde{\mathbf{f}}}\ \mathcal{L}_{\widetilde{\mathbf{g}}}$ $\inf_{\mathbf{u} \in \mathbb{R}^m} \dot{\mathtt{C}}(\mathbf{x}, \mathbf{u}) \leq -\alpha(\mathtt{C}(\mathbf{x}))$ $\dot{\mathtt{C}}(\mathbf{x}, \mathbf{u}) = \nabla \mathtt{C}(\mathbf{x})(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u})$

True Dynamics

Physical Robot

CCF Derivative Uncertainty

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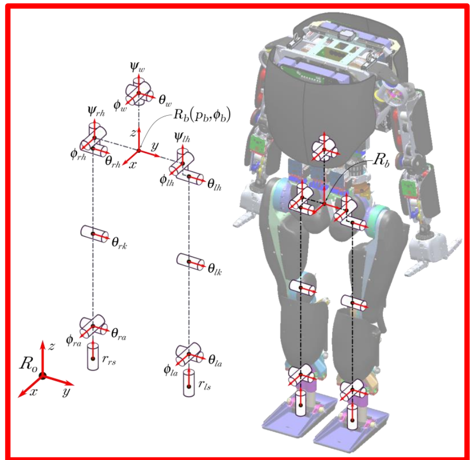
CCF Derivative Uncertainty



$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\pm\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right)$$

CCF Derivative Uncertainty

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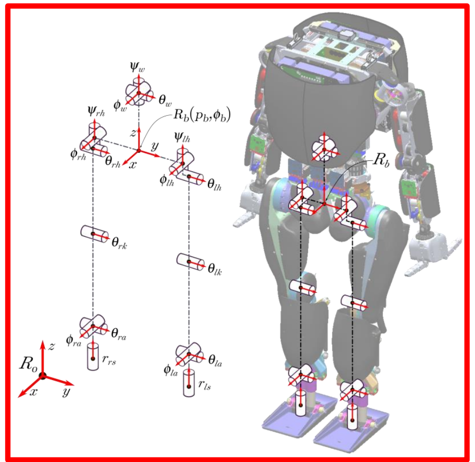
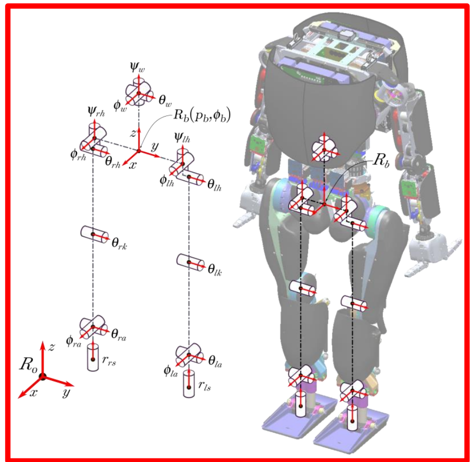


$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\pm\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right)$$

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{(\mathbf{g}(\mathbf{x}) - \widehat{\mathbf{g}}(\mathbf{x}))}_{\widetilde{\mathbf{g}}(\mathbf{x})}\mathbf{u} + \underbrace{\mathbf{f}(\mathbf{x}) - \widehat{\mathbf{f}}(\mathbf{x})}_{\widetilde{\mathbf{f}}(\mathbf{x})}$$

CCF Derivative Uncertainty



$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\pm\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right)$$

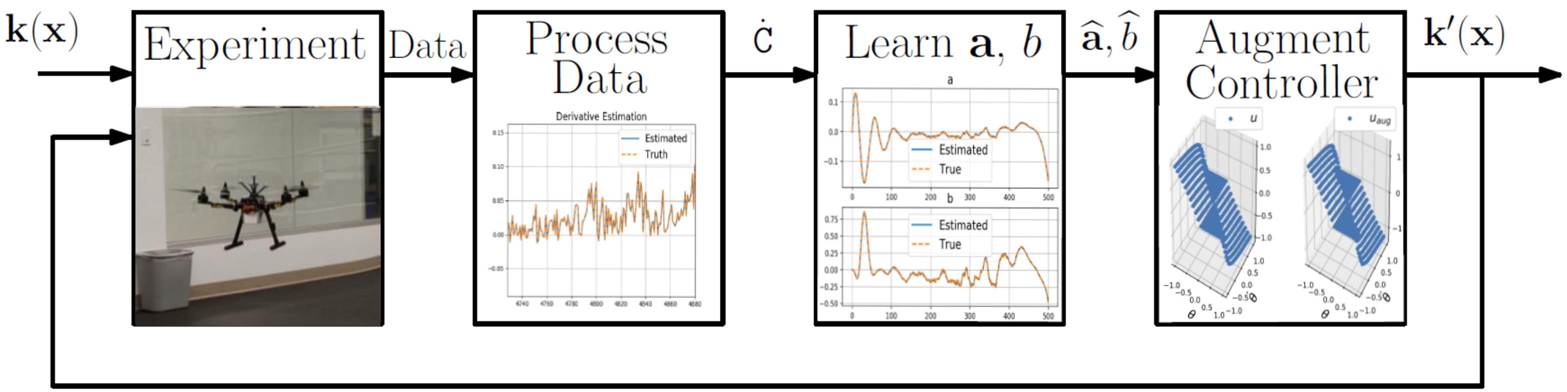
$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{\left(\mathbf{g}(\mathbf{x}) - \widehat{\mathbf{g}}(\mathbf{x})\right)}_{\widetilde{\mathbf{g}}(\mathbf{x})}\mathbf{u} + \underbrace{\mathbf{f}(\mathbf{x}) - \widehat{\mathbf{f}}(\mathbf{x})}_{\widetilde{\mathbf{f}}(\mathbf{x})}$$

$$\dot{C}(\mathbf{x},\mathbf{u}) = \underbrace{\nabla C(\mathbf{x})\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right)}_{\widehat{\dot{C}}(\mathbf{x},\mathbf{u})} + \underbrace{\nabla C(\mathbf{x})\widetilde{\mathbf{g}}(\mathbf{x})}_{\mathbf{a}(\mathbf{x})^\top}\mathbf{u} + \underbrace{\nabla C(\mathbf{x})\widetilde{\mathbf{f}}(\mathbf{x})}_{b(\mathbf{x})}$$

Learning CCF Derivatives

CCF Derivative Estimator

$$\dot{\mathtt{C}}(\mathbf{x},\mathbf{u}) \approx \widehat{\dot{\mathtt{C}}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x})$$

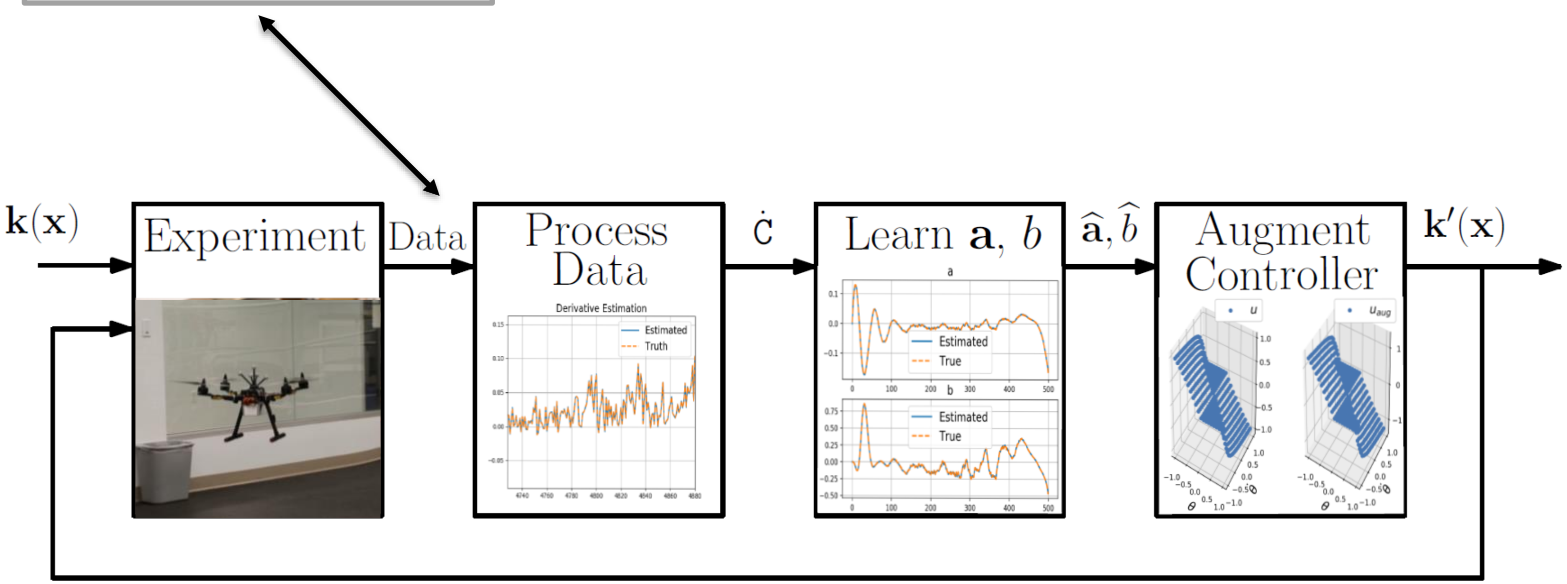


[13] A. J. Taylor, V. D. Dorobantu, et al., "Episodic Learning with Control Lyapunov Functions for Uncertain Robotic Systems", 2019.
[14] A. J. Taylor, V. D. Dorobantu, et al., "A Control Lyapunov Perspective on Episodic Learning via Projection to State Stability", 2019.
[15] A. J. Taylor, et al., "Learning for Safety-Critical Control with Control Barrier Functions", 2020.
[16] A. J. Taylor, et al., "A Control Barrier Perspective on Episodic Learning via Projection-to-State Safety", 2020.

Learning CCF Derivatives

Dataset

$D = \{((\mathbf{x}_i, \mathbf{u}_i), \dot{\mathtt{c}}_i)\}_{i=1}^{N}$



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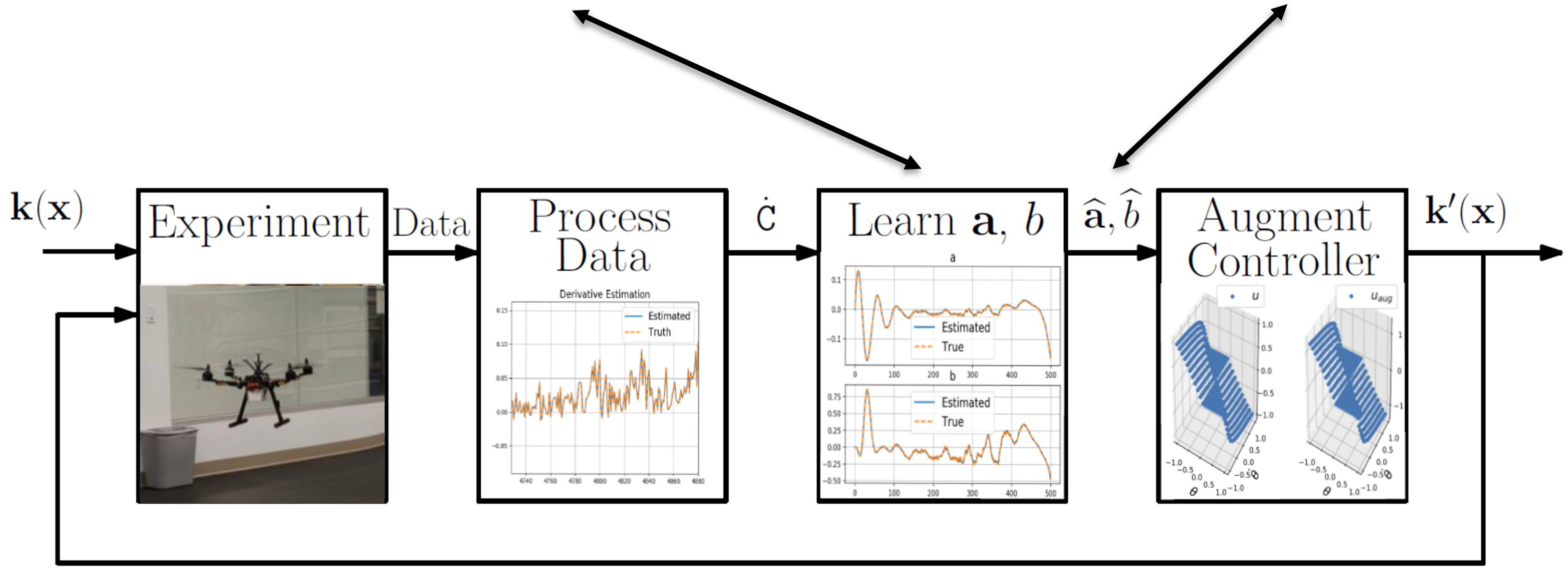
Learning CCF Derivatives

Model Class

$$\widehat{\mathbf{a}} \in \mathcal{H}_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^m$$
$$\widehat{b} \in \mathcal{H}_b : \mathbb{R}^n \to \mathbb{R}$$

Regression

$$\min_{\substack{\widehat{\mathbf{a}} \in \mathcal{H}_{\mathbf{a}} \\ \widehat{b} \in \mathcal{H}_b}} \sum_{i=1}^{N} \mathcal{L}\left(\widehat{\dot{\mathtt{C}}}(\mathbf{x}_i, \mathbf{u}_i) + \widehat{\mathbf{a}}(\mathbf{x}_i)^\top \mathbf{u}_i + \widehat{b}(\mathbf{x}_i), \dot{\mathtt{C}}_i\right)$$



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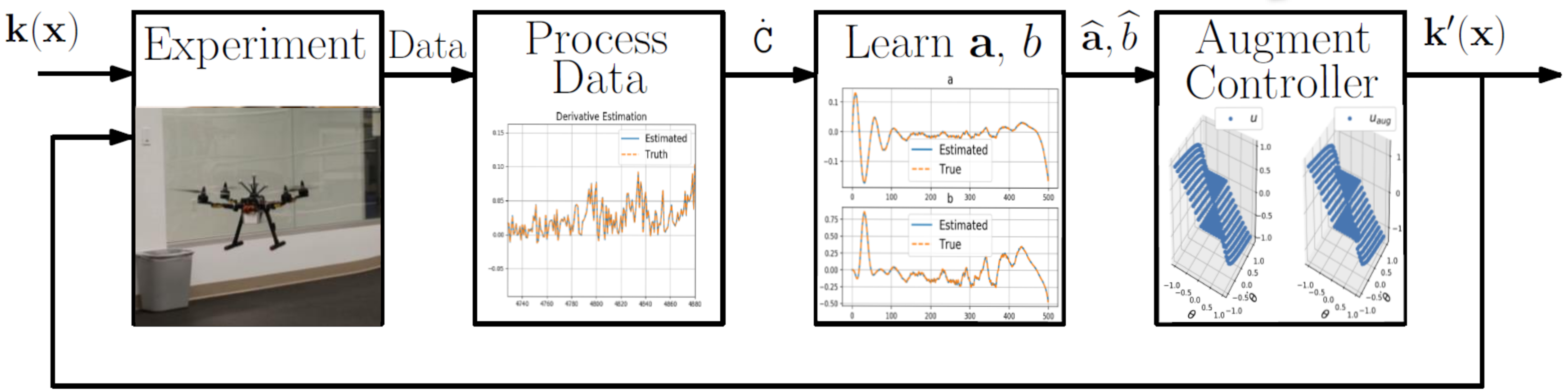
Learning CCF Derivatives

Controller Design

$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_d(\mathbf{x})\|_2^2$$
$$\text{s.t.} \quad \widehat{\dot{\mathtt{C}}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \leq -\alpha(\mathtt{C}(\mathbf{x}))$$



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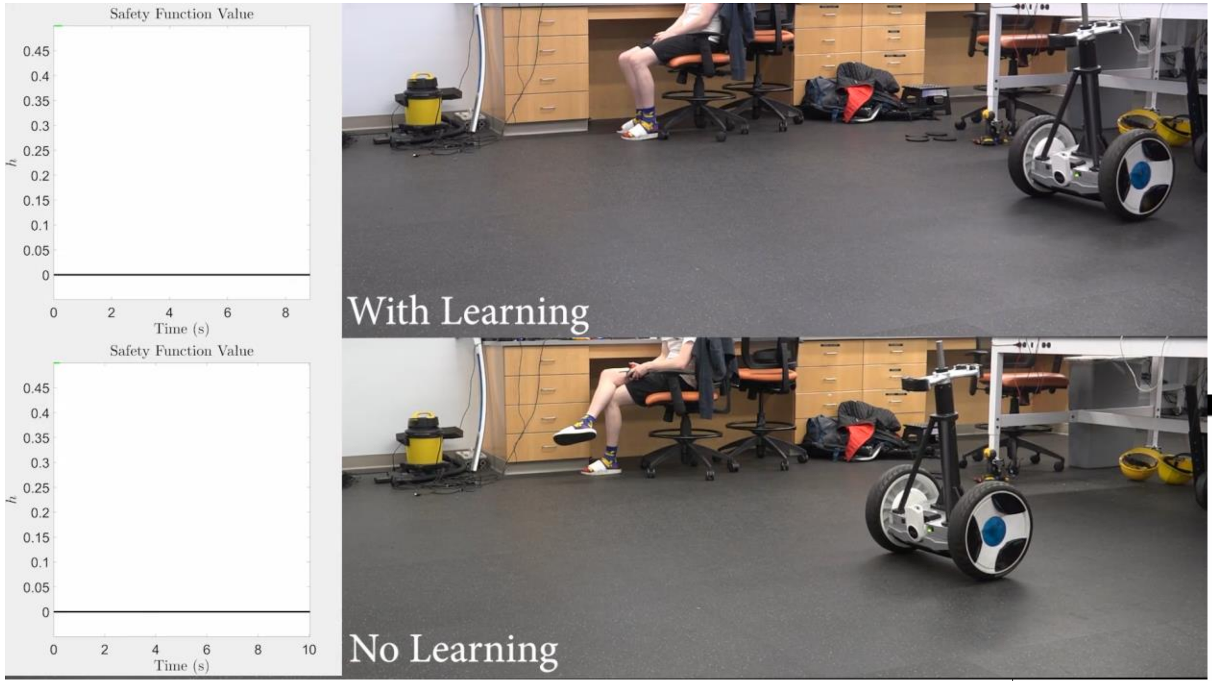
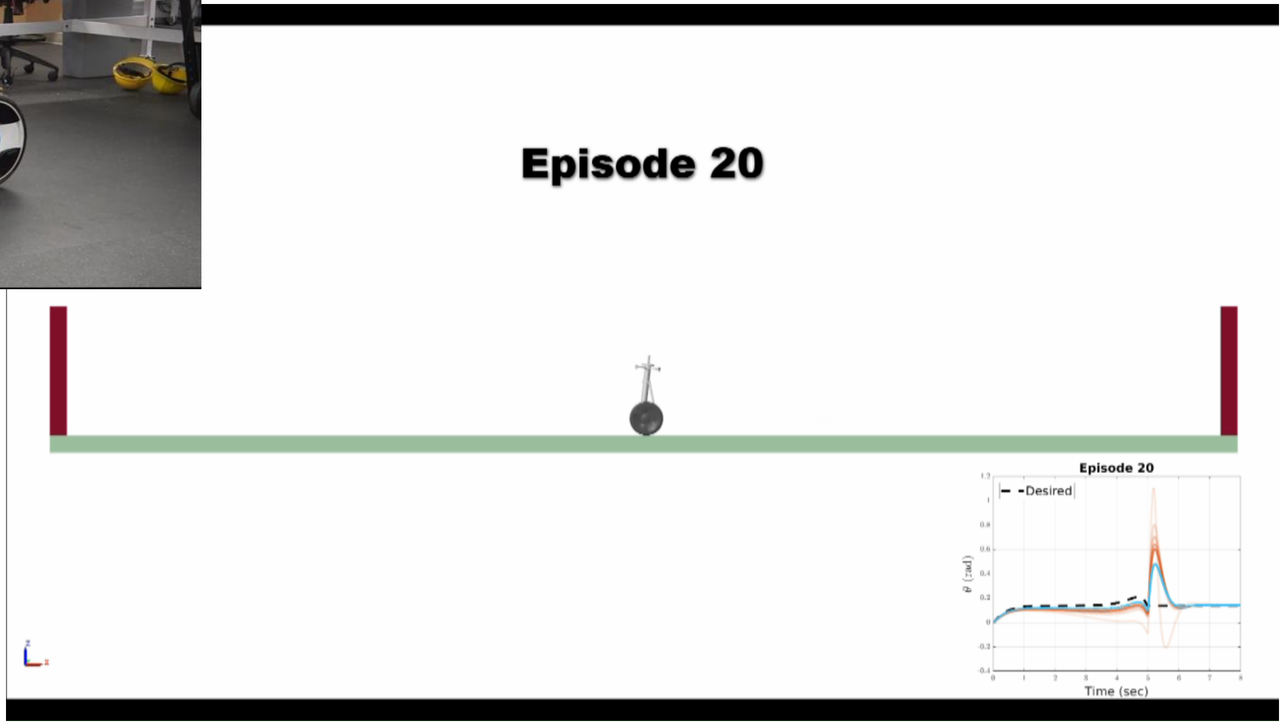
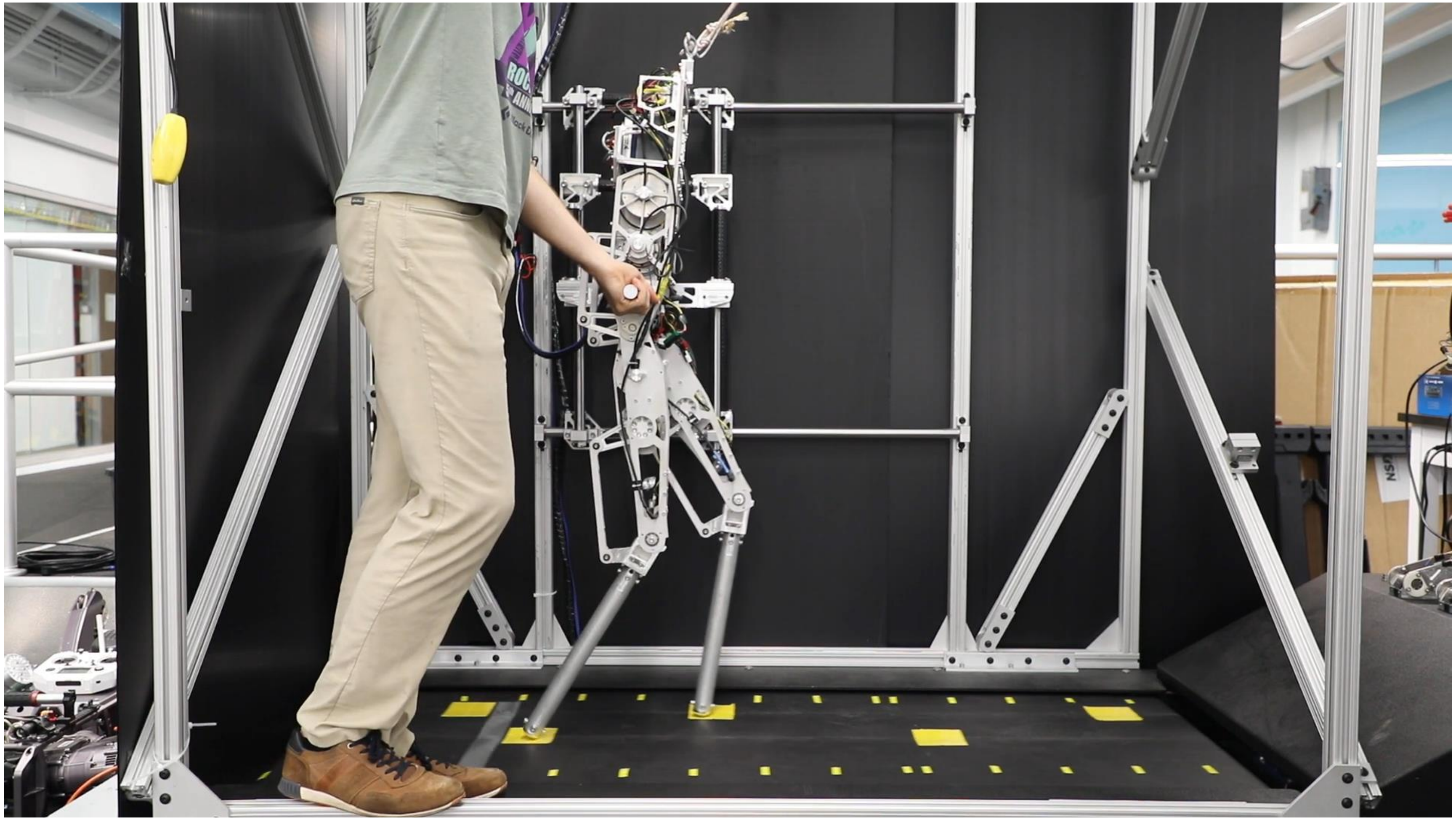
Learning CCF Derivatives





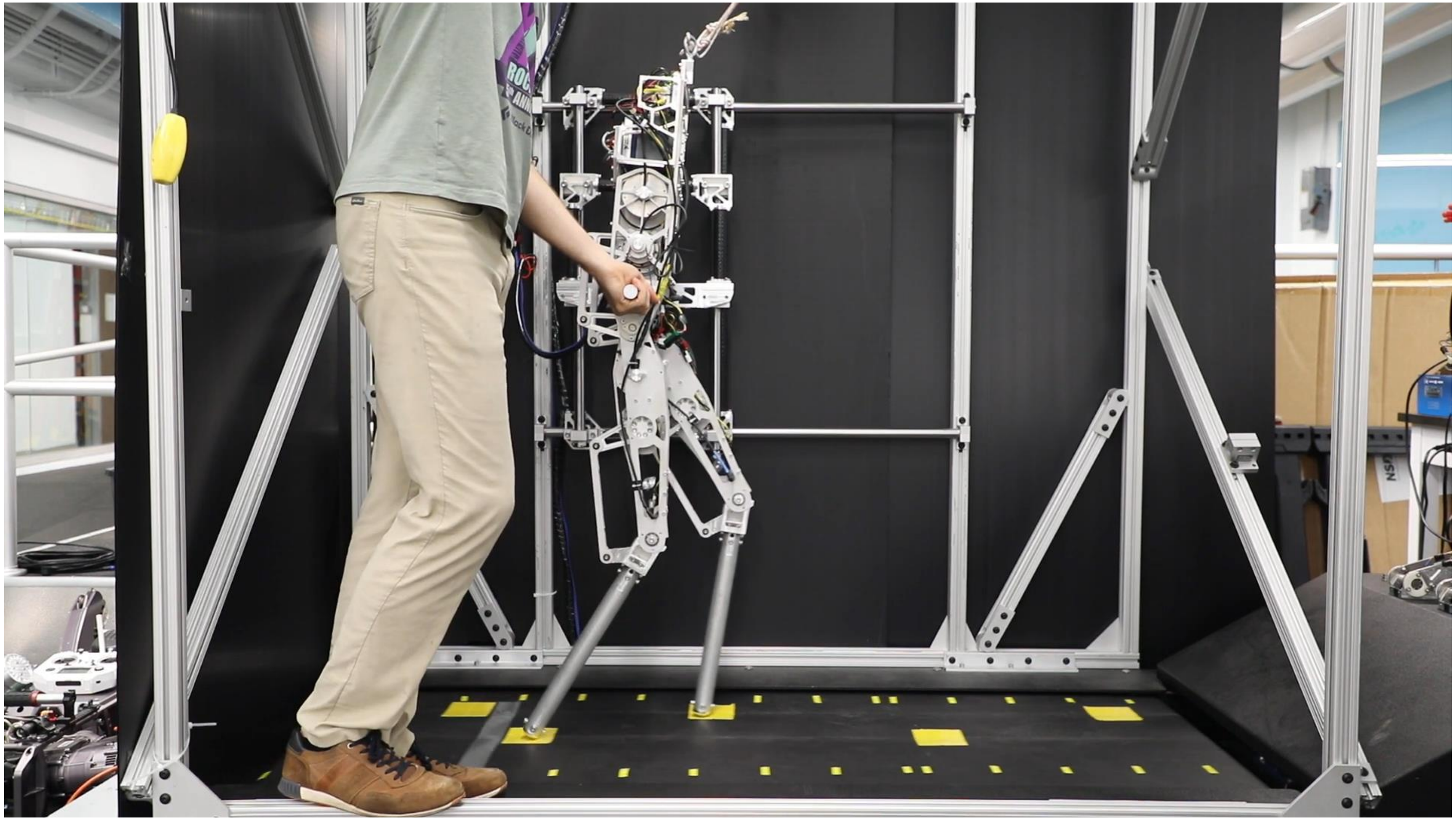
Learning CCF Derivatives



[17] N. Csomay-Shanklin, R. K. Cosner, M. Dai, **A. J. Taylor**, A. D. Ames, "Episodic Learning for Safe Bipedal Locomotion with Control Barrier Functions and Projection-to-State Safety", 2021.

Learning CCF Derivatives

Poor generalization of actuation



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Actuation Characterization

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Estimator Error
$\widetilde{\dot{\mathtt{C}}}_i = \dot{\mathtt{C}}_i - \widehat{\dot{\mathtt{C}}}(\mathbf{x}_i, u_i) = a(\mathbf{x}_i)u_i + b(\mathbf{x}_i)$

Actuation Characterization

Estimator Error

$$\widetilde{\dot{\mathtt{C}}}_i = \dot{\mathtt{C}}_i - \widehat{\dot{\mathtt{C}}}(\mathbf{x}_i, u_i) = a(\mathbf{x}_i)u_i + b(\mathbf{x}_i)$$

Affine Form

$$\widetilde{\dot{\mathtt{C}}}_i = [u_i \ \ 1] \begin{bmatrix} a(\mathbf{x}_i) \\ b(\mathbf{x}_i) \end{bmatrix}$$

Actuation Characterization

Estimator Error

$$\tilde{\dot{\mathrm{C}}}_i = \dot{\mathrm{C}}_i - \widehat{\dot{\mathrm{C}}}(\mathbf{x}_i, u_i) = a(\mathbf{x}_i)u_i + b(\mathbf{x}_i)$$

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Decomposition

$$\begin{bmatrix} a(\mathbf{x}_i) \\ b(\mathbf{x}_i) \end{bmatrix} = \gamma_1 \begin{bmatrix} u_i \\ 1 \end{bmatrix} + \gamma_2 \begin{bmatrix} -\frac{1}{u_i} \\ 1 \end{bmatrix}$$

Actuation Characterization

Estimator Error

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Information Content

$$\tilde{\dot{\mathtt{C}}}_i = \gamma_1 \left\| \begin{bmatrix} u_i \\ 1 \end{bmatrix} \right\|_2^2$$

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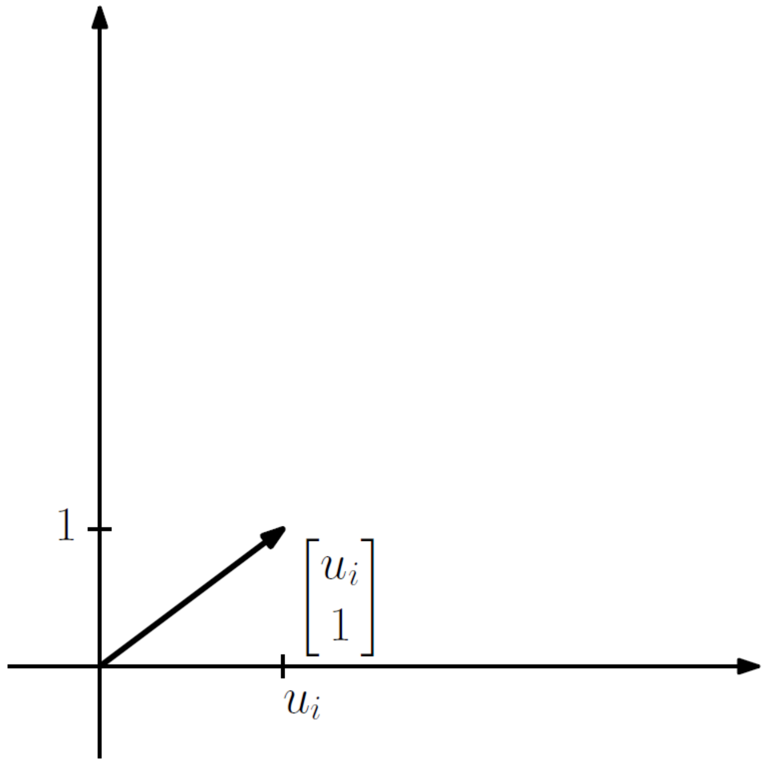

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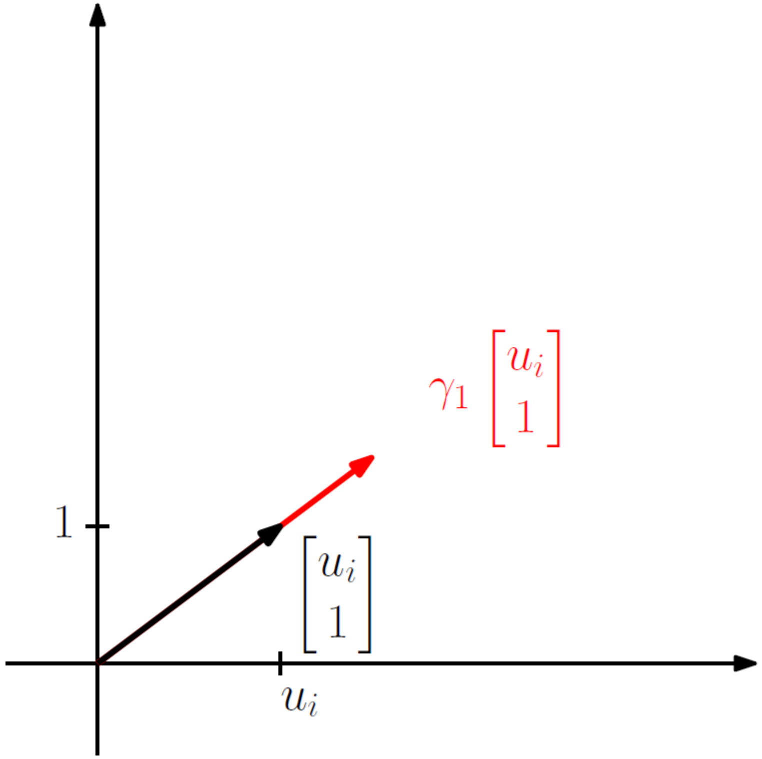

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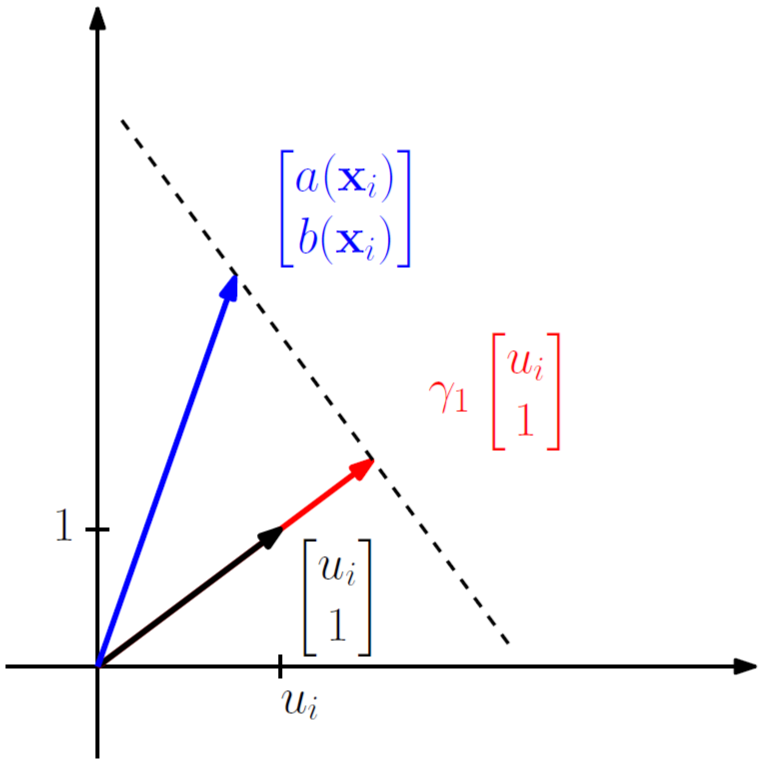

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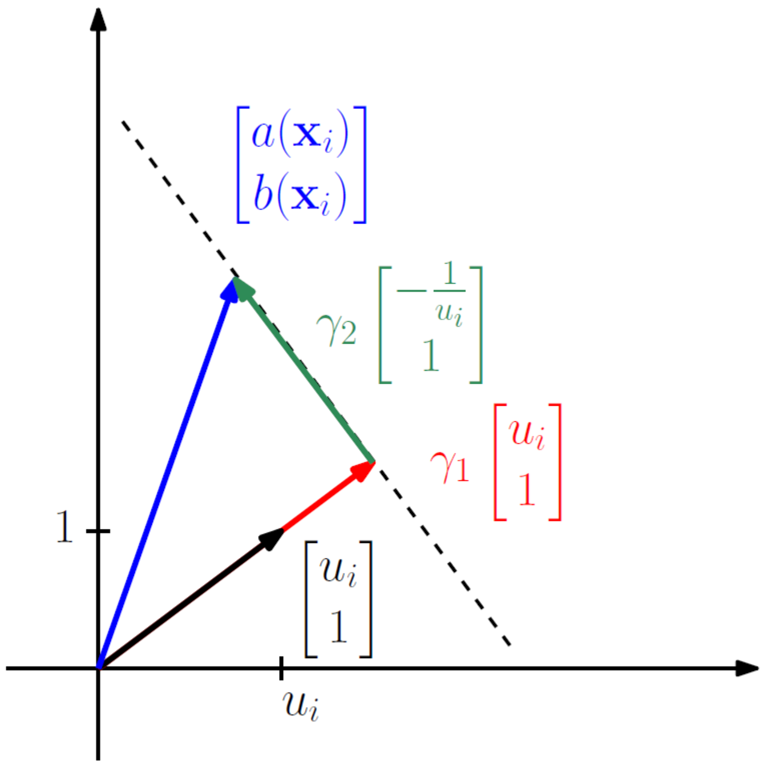
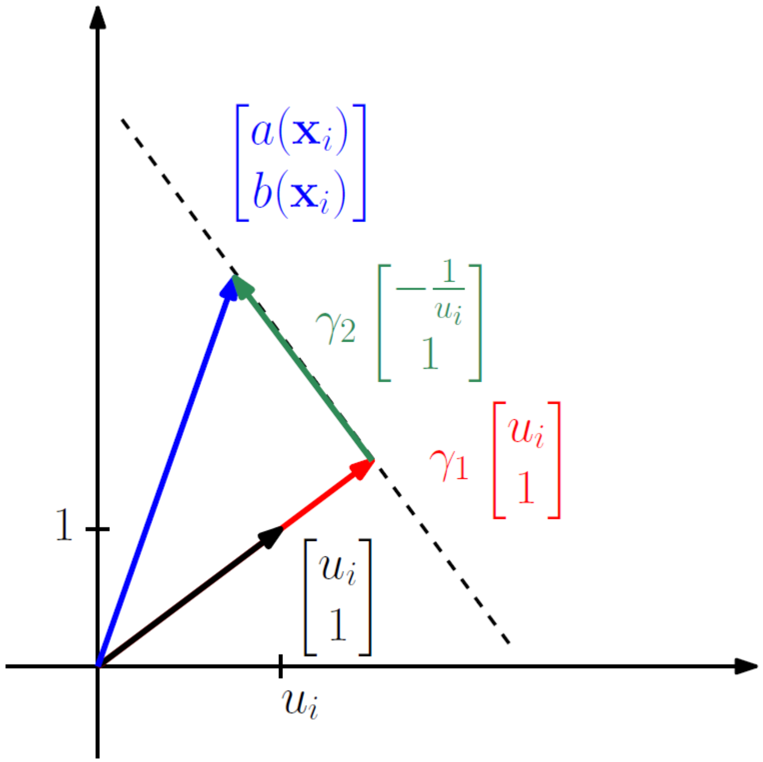


Actuation Characterization

Learning Goal

$$\widetilde{\dot{c}}_i \approx \widehat{a}(\mathbf{x}_i)u_i + \widehat{b}(\mathbf{x}_i)$$



Actuation Characterization

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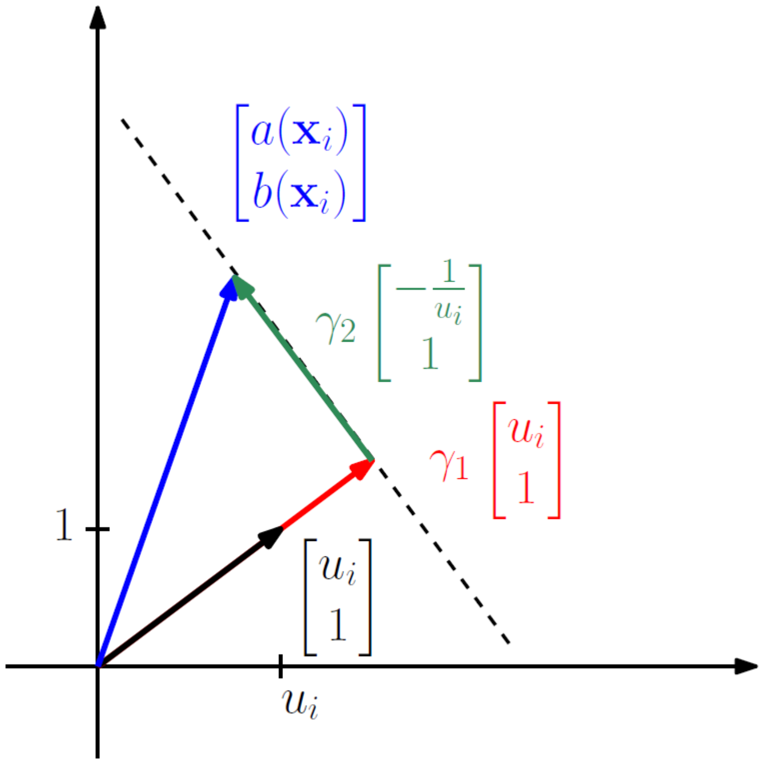

Actuation Characterization

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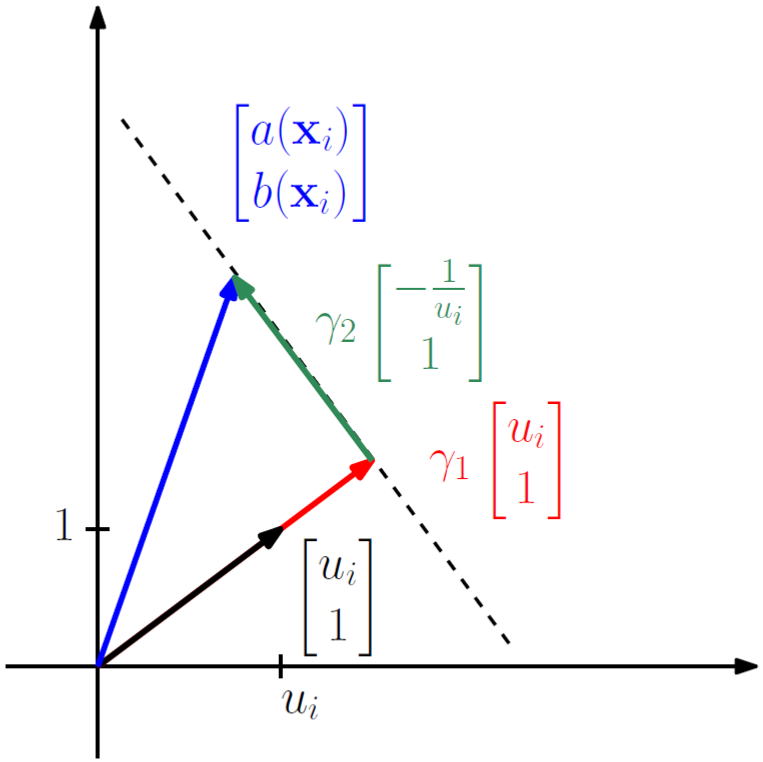

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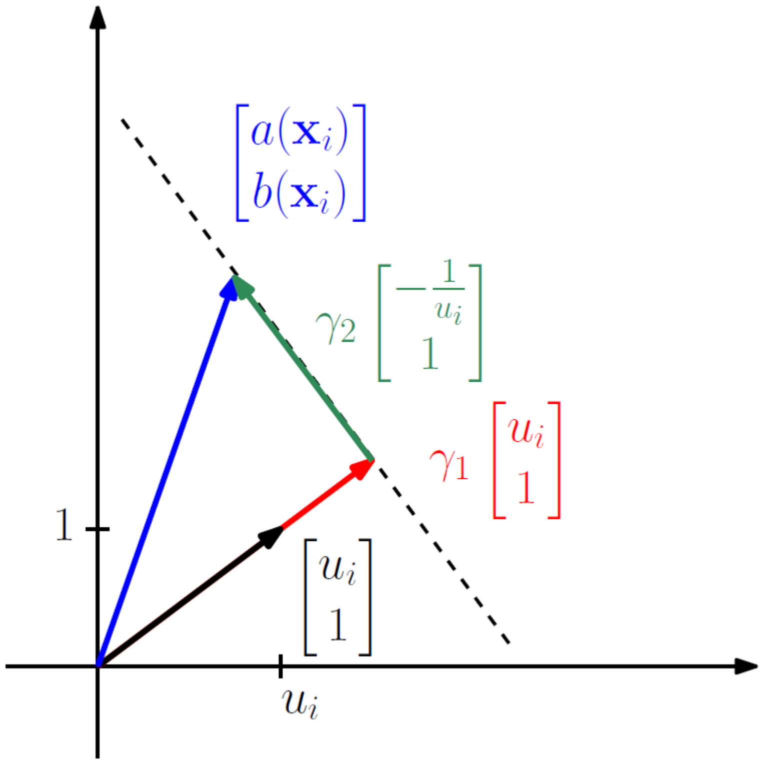
Learning Outcome

$$\widehat{\gamma}_1 \approx \gamma_1$$



Actuation Characterization

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Learning Goal

$$\widetilde{\dot{C}}_i \approx \widehat{a}(\mathbf{x}_i)u_i + \widehat{b}(\mathbf{x}_i)$$

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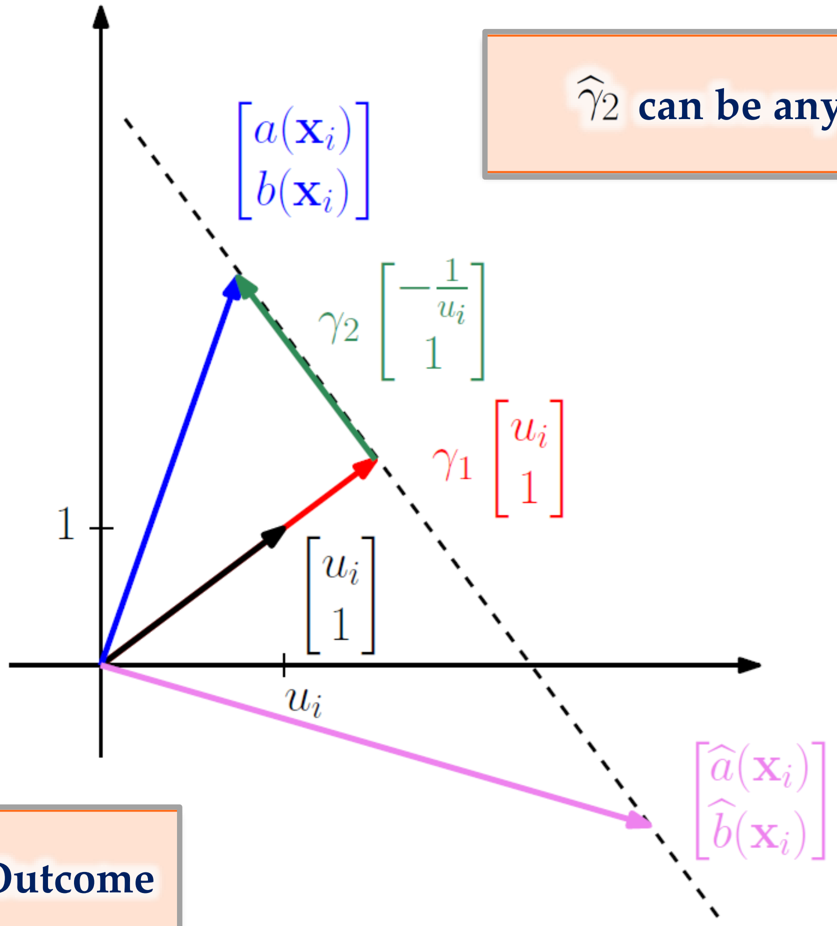

$\widehat{\gamma}_2$ can be anything!

Actuation Characterization

Learning Goal

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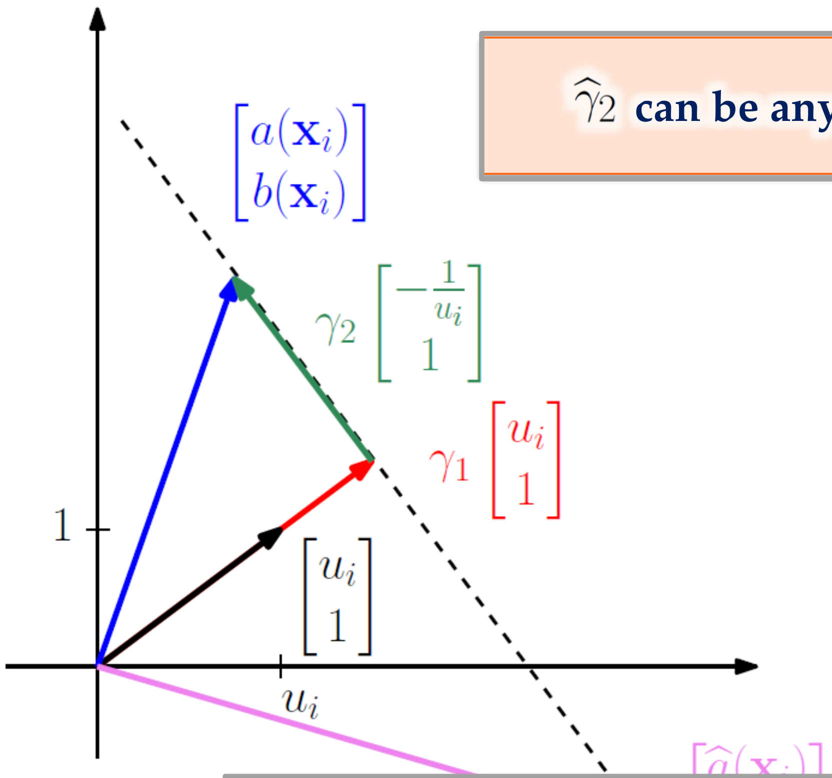

$\widehat{\gamma}_2$ can be anything!

Poor Controller Design

$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_d(\mathbf{x})\|_2^2$$
$$\text{s.t.} \quad \dot{C}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \leq -\alpha(C(\mathbf{x}))$$

More Data?

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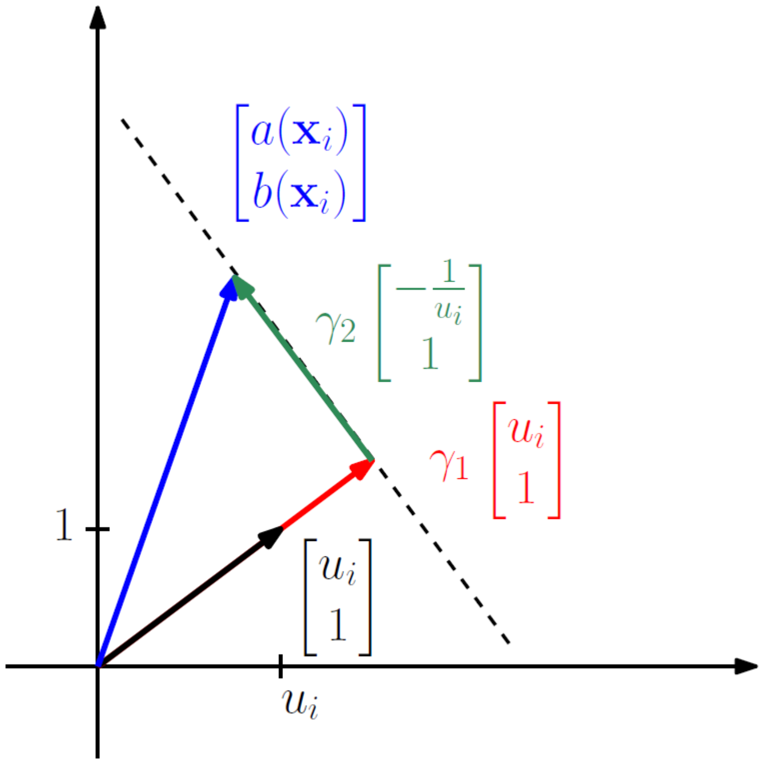
Additional Data

$$\widetilde{\dot{c}}_j \approx \widehat{a}(\mathbf{x}_i)u_j + \widehat{b}(\mathbf{x}_i)$$



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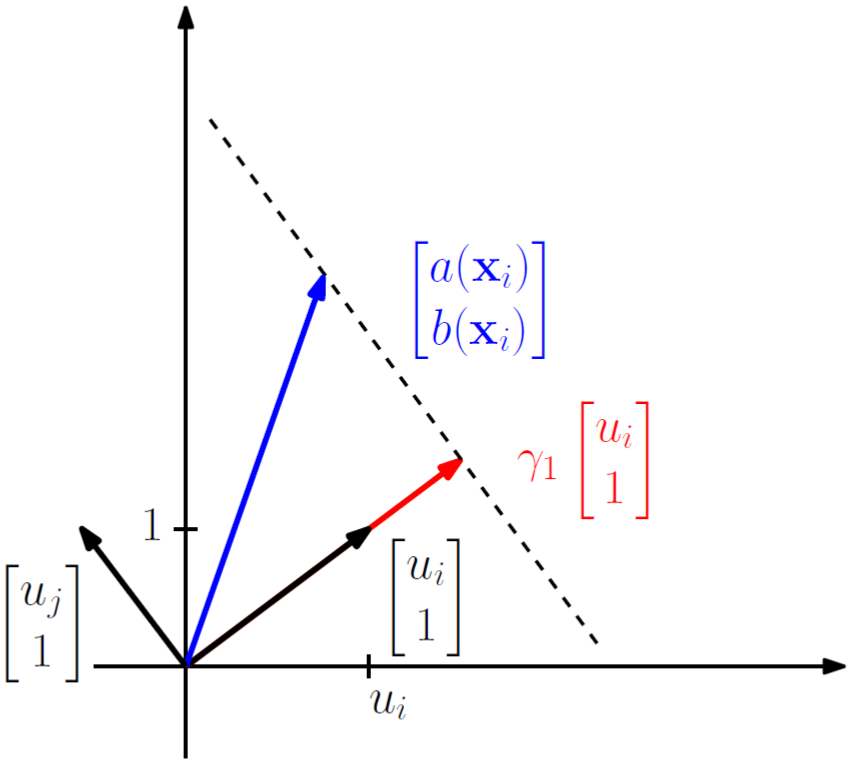

More Data?

Additional Data
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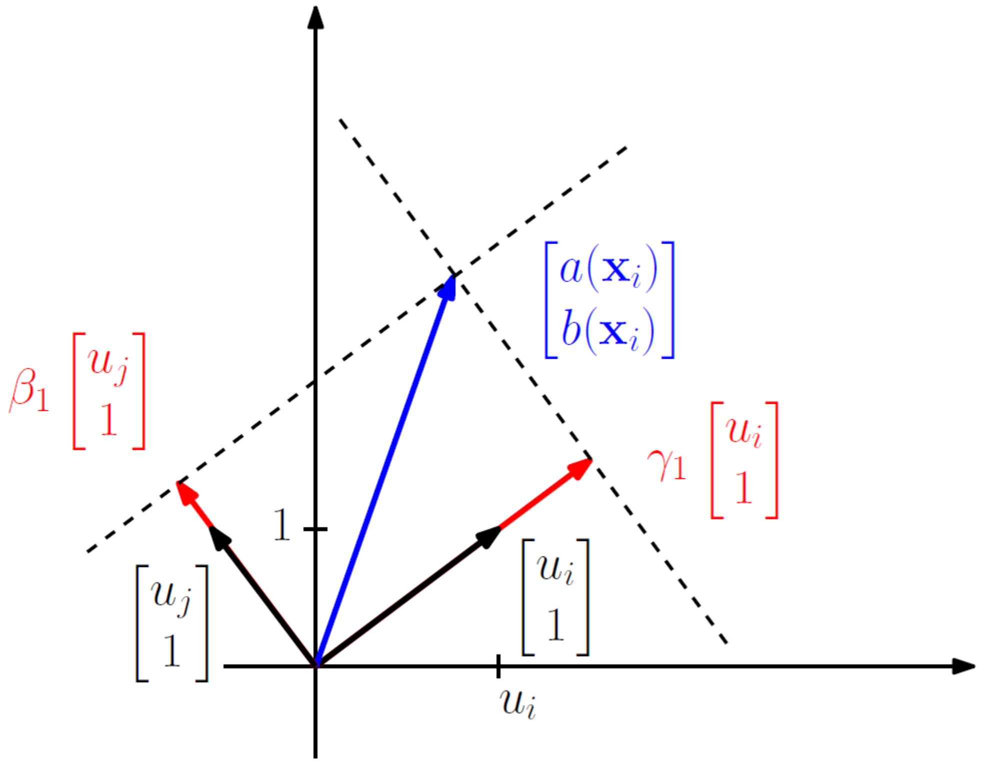

More Data?

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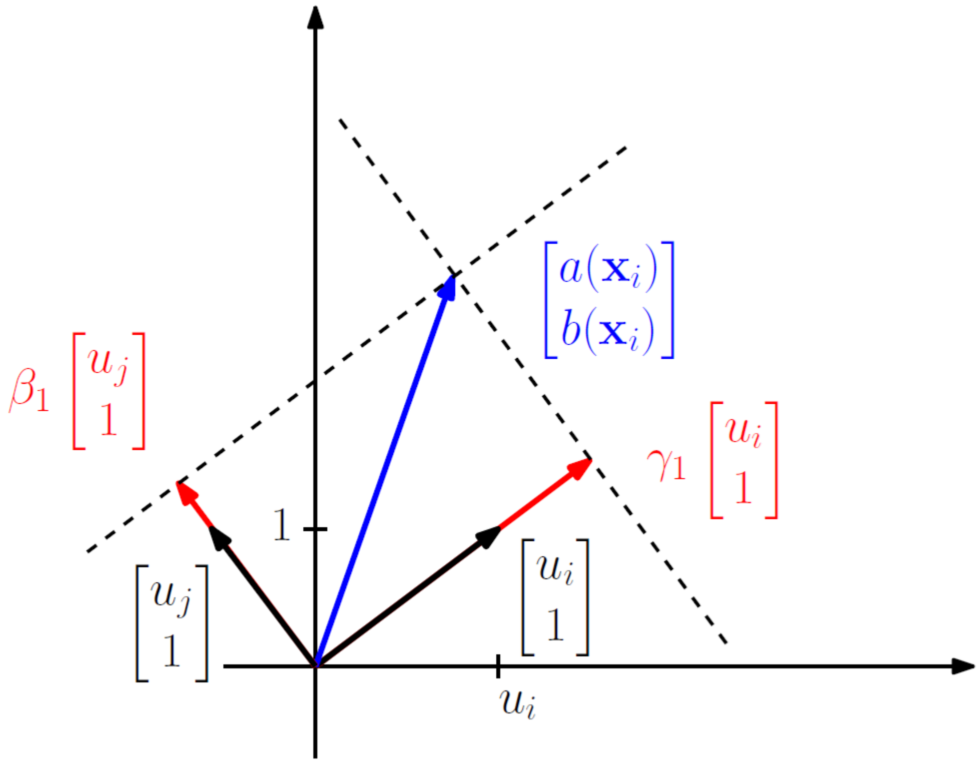
Additional Data

$$\tilde{\dot{\mathrm{c}}}_j \approx \widehat{a}(\mathbf{x}_i)u_j + \widehat{b}(\mathbf{x}_i)$$

Least Squares

$$\begin{bmatrix}\tilde{\dot{\mathrm{c}}}_i \\ \tilde{\dot{\mathrm{c}}}_j\end{bmatrix} \approx \underbrace{\begin{bmatrix}u_i & 1 \\ u_j & 1\end{bmatrix}}_{\mathbf{A}} \begin{bmatrix}\widehat{a}(\mathbf{x}_i) \\ \widehat{b}(\mathbf{x}_i)\end{bmatrix}$$



More Data?

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Additional Data

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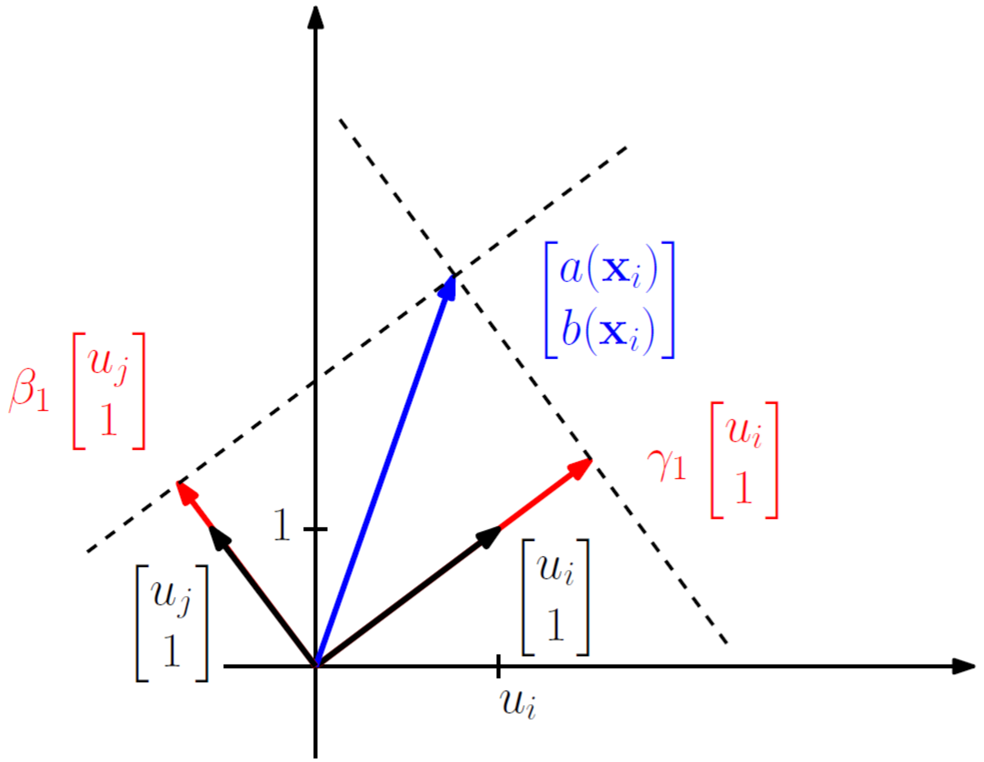
$$\begin{bmatrix} \tilde{\dot{\mathrm{C}}}_i \\ \tilde{\dot{\mathrm{C}}}_j \end{bmatrix} \approx \underbrace{\begin{bmatrix} u_i & 1 \\ u_j & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \widehat{a}(\mathbf{x}_i) \\ \widehat{b}(\mathbf{x}_i) \end{bmatrix}$$

Different Inputs

$$u_j = -u_i$$



More Data?

Additional Data

$$\widetilde{\dot{\text{C}}}_j \approx \widehat{a}(\mathbf{x}_i)u_j + \widehat{b}(\mathbf{x}_i)$$

Least Squares

$$\begin{bmatrix} \widetilde{\dot{\text{C}}}_i \\ \widetilde{\dot{\text{C}}}_j \end{bmatrix} \approx \underbrace{\begin{bmatrix} u_i & 1 \\ u_j & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \widehat{a}(\mathbf{x}_i) \\ \widehat{b}(\mathbf{x}_i) \end{bmatrix}$$

Different Inputs

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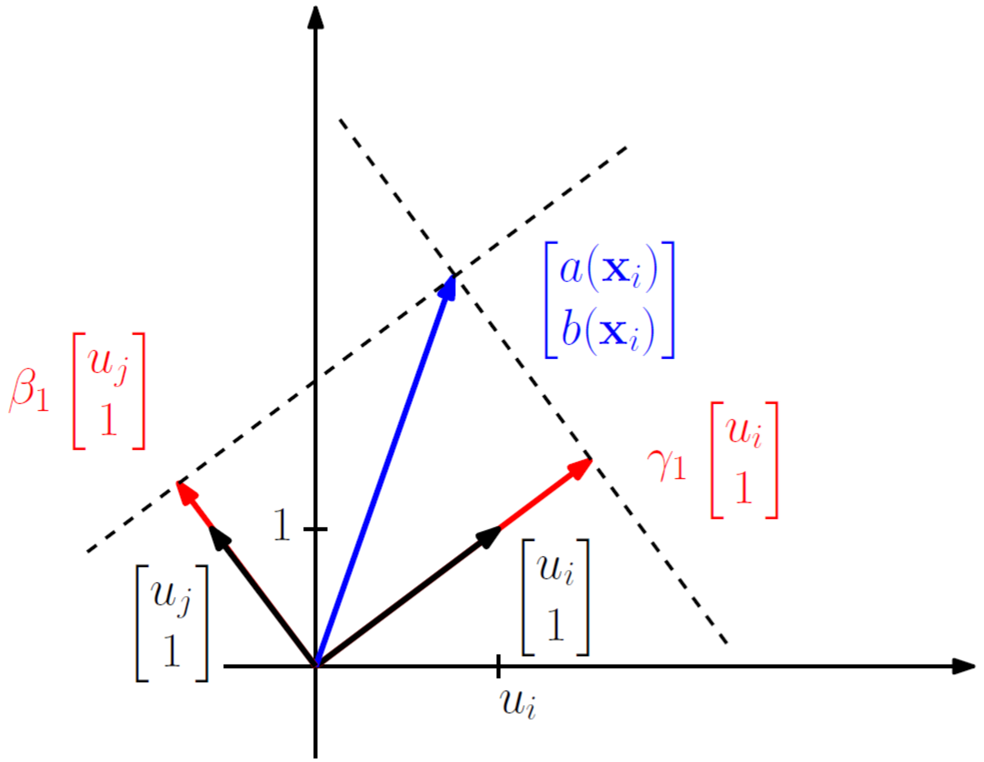
Conditioning

$$\text{cond}(\mathbf{A}) = 1$$



More Data?

Additional Data

$$\tilde{\dot{\mathrm{c}}}_j \approx \widehat{a}(\mathbf{x}_i)u_j + \widehat{b}(\mathbf{x}_i)$$

Least Squares

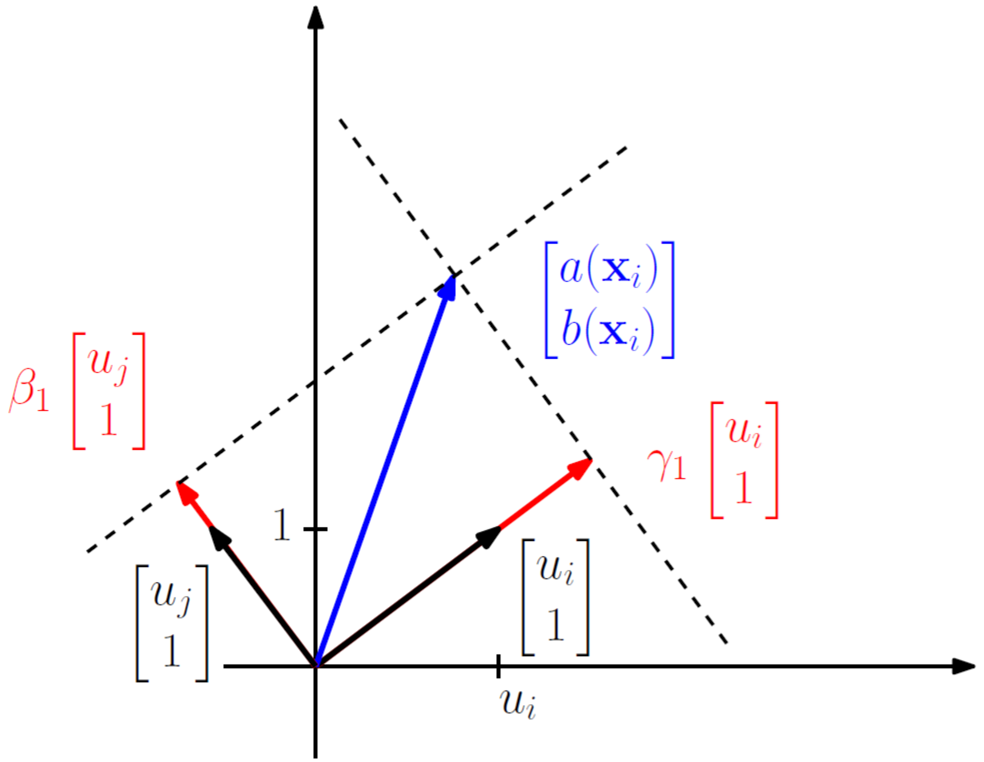
$$\begin{bmatrix} \tilde{\dot{\mathrm{c}}}_i \\ \tilde{\dot{\mathrm{c}}}_j \end{bmatrix} \approx \underbrace{\begin{bmatrix} u_i & 1 \\ u_j & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \widehat{a}(\mathbf{x}_i) \\ \widehat{b}(\mathbf{x}_i) \end{bmatrix}$$

Different Inputs

$$u_j = -u_i$$

Conditioning

$$\mathrm{cond}(\mathbf{A}) = 1$$



Collecting this input data may be infeasible at $\mathbf{x}_i$!

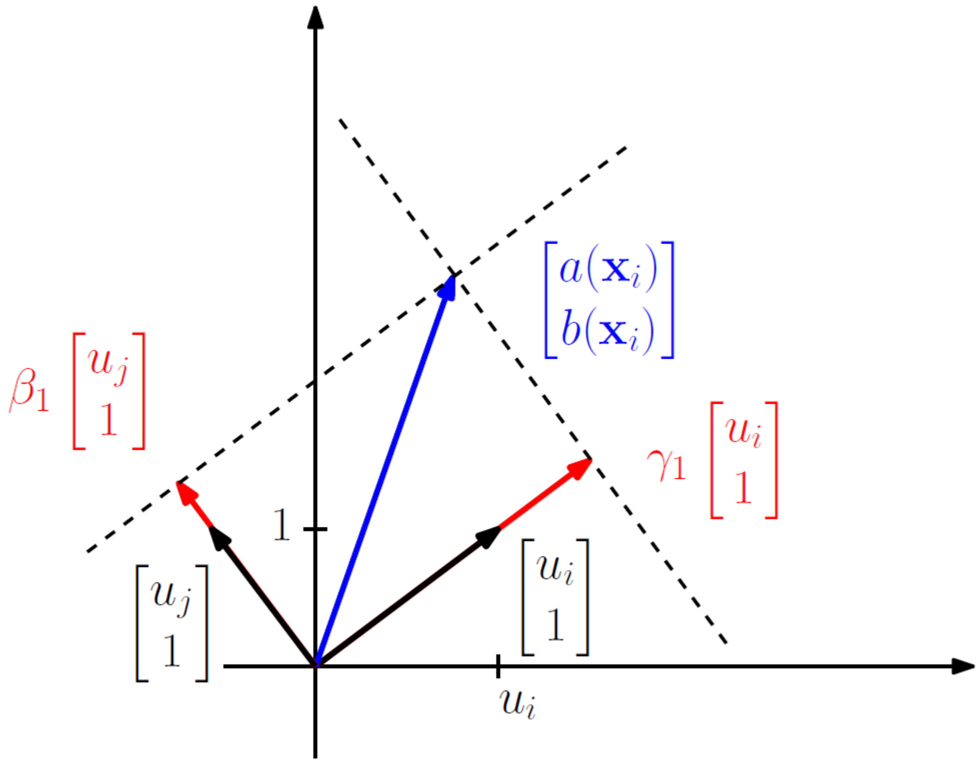
More Data?

Additional Data

$$\tilde{\dot{\mathrm{c}}}_j \approx \widehat{a}(\mathbf{x}_i)u_j + \widehat{b}(\mathbf{x}_i)$$

Least Squares

$$\begin{bmatrix} \tilde{\dot{\mathrm{c}}}_i \\ \tilde{\dot{\mathrm{c}}}_j \end{bmatrix} \approx \underbrace{\begin{bmatrix} u_i & 1 \\ u_j & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \widehat{a}(\mathbf{x}_i) \\ \widehat{b}(\mathbf{x}_i) \end{bmatrix}$$



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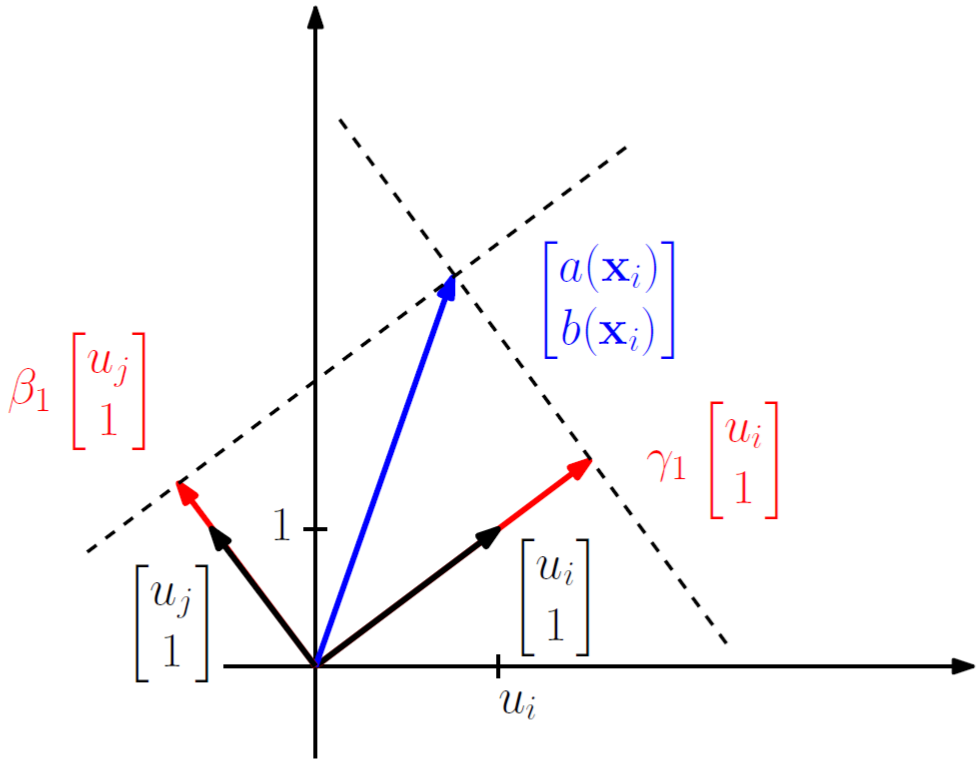
Similar Inputs

$$u_j = u_i + \epsilon$$



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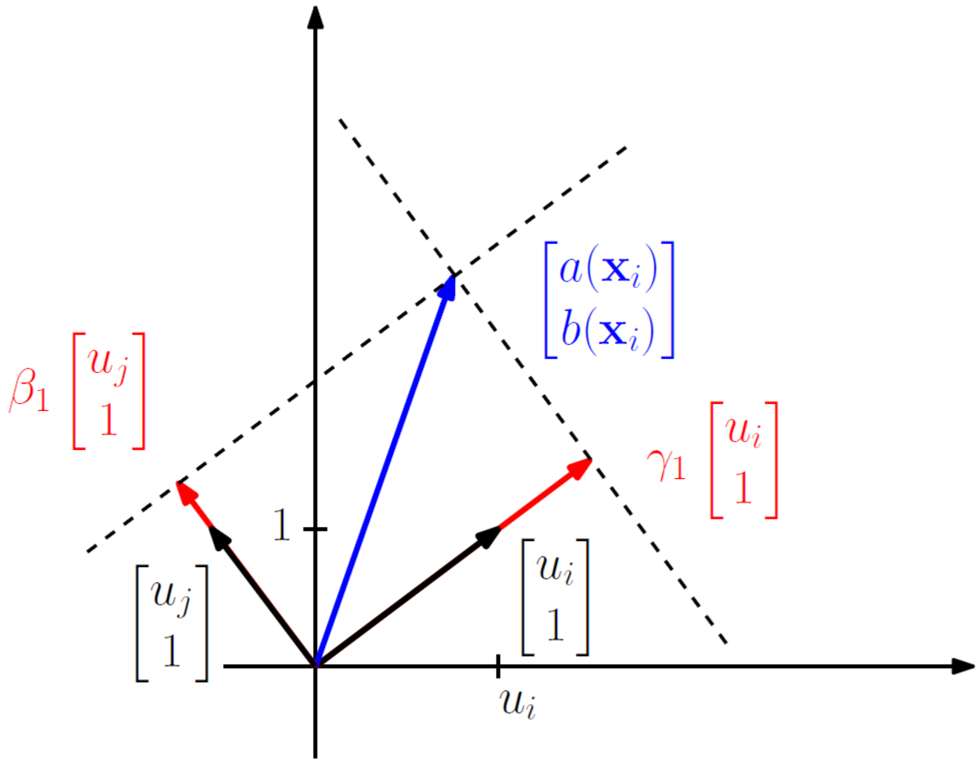
Similar Inputs

$$u_j = u_i + \epsilon$$

Conditioning

$$\lim_{\epsilon \to 0} \mathrm{cond}(\mathbf{A}) = \infty$$



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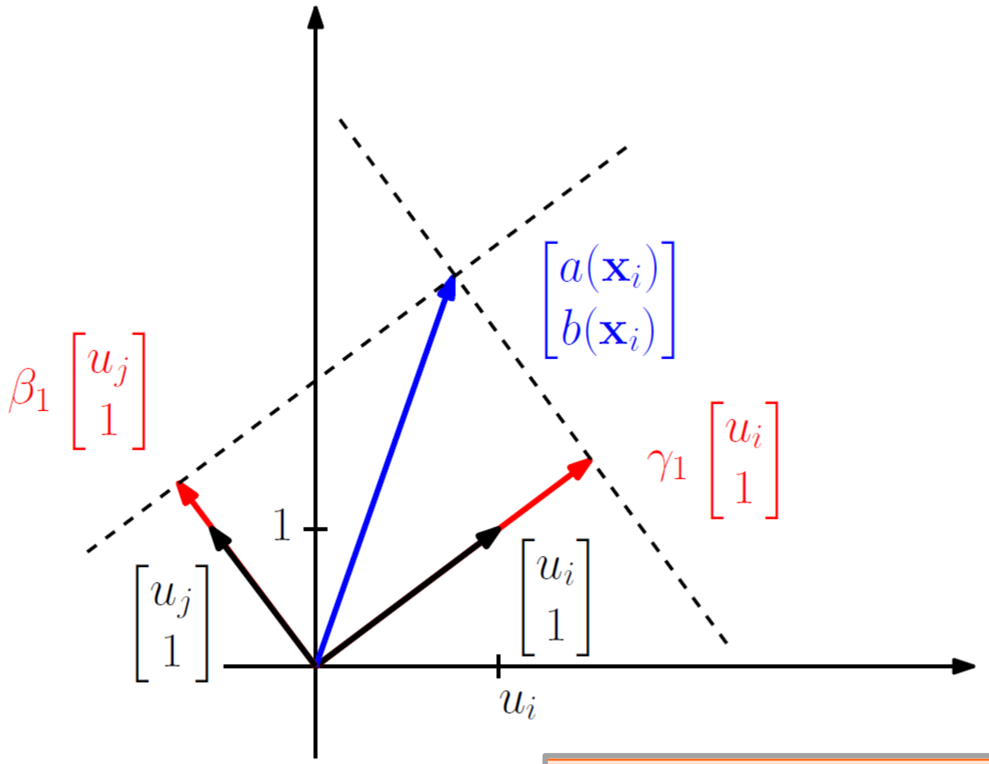

Similar inputs are insufficient to fully characterize actuation!

More Data?

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$$\widetilde{\dot{\mathrm{c}}}_j \approx \widehat{a}(\mathbf{x}_i)u_j + \widehat{b}(\mathbf{x}_i)$$

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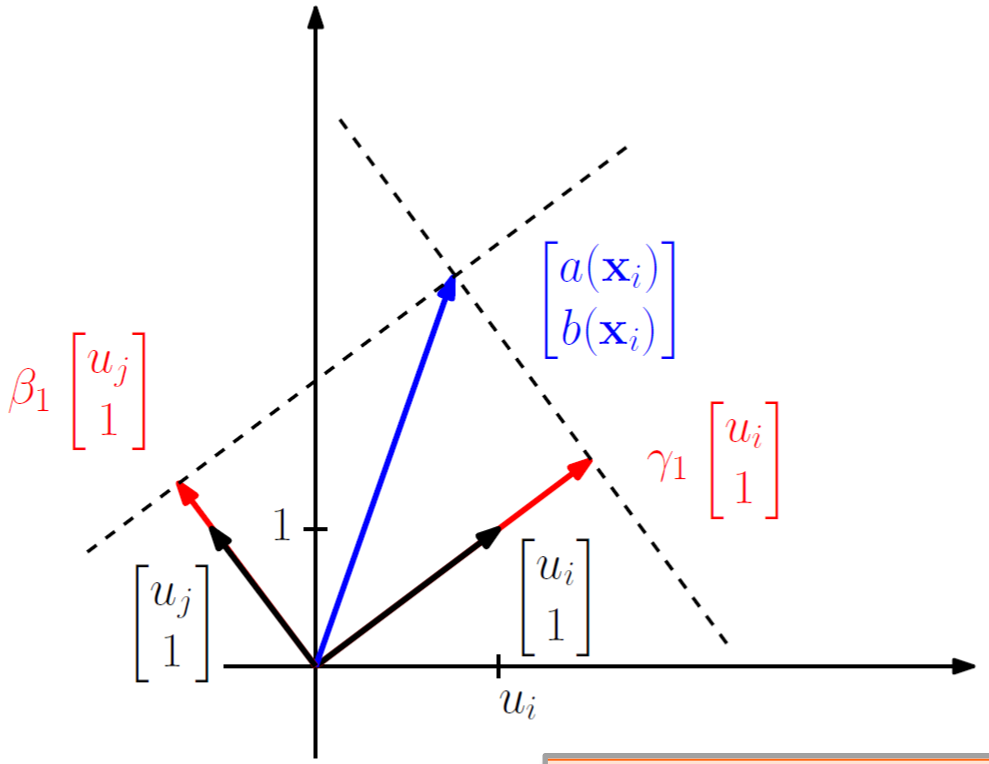

Similar inputs are insufficient to fully characterize actuation!

More inputs require more directions of input data!

More Data?

How do we work with a partial characterization of actuation?

Additional Data

$$\widetilde{\dot{\mathrm{C}}}_j \approx \widehat{a}(\mathbf{x}_i)u_j + \widehat{b}(\mathbf{x}_i)$$

Least Squares

$$\begin{bmatrix} \widetilde{\dot{\mathrm{C}}}_i \\ \widetilde{\dot{\mathrm{C}}}_j \end{bmatrix} \approx \underbrace{\begin{bmatrix} u_i & 1 \\ u_j & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \widehat{a}(\mathbf{x}_i) \\ \widehat{b}(\mathbf{x}_i) \end{bmatrix}$$

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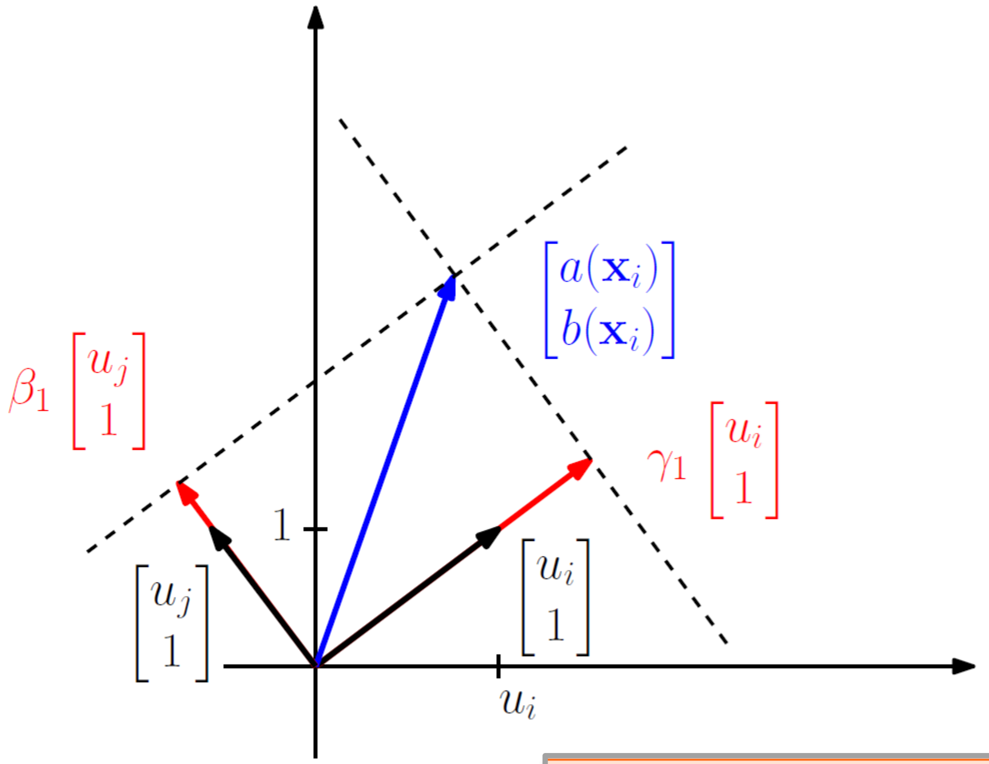
Conditioning

$$\lim_{\epsilon \to 0} \mathrm{cond}(\mathbf{A}) = \infty$$



Similar inputs are insufficient to fully characterize actuation!

More inputs require more directions of input data!

Uncertainty Sets

Dataset
$D = \{(\mathbf{x}_i, \mathbf{u}_i, \dot{\mathbf{x}}_i)\}_{i=1}^N$

Uncertainty Sets

Dataset

$$D = \{(\mathbf{x}_i, \mathbf{u}_i, \dot{\mathbf{x}}_i)\}_{i=1}^N$$

State Dynamics Error

$$\widetilde{\mathbf{F}}_i = \dot{\mathbf{x}}_i - (\widehat{\mathbf{f}}(\mathbf{x}_i) + \widehat{\mathbf{g}}(\mathbf{x}_i)\mathbf{u}_i)$$

$$\widetilde{\mathbf{F}}_i = \widetilde{\mathbf{f}}(\mathbf{x}_i) + \widetilde{\mathbf{g}}(\mathbf{x}_i)\mathbf{u}_i$$

Uncertainty Sets

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Test Point

$$\widetilde{\mathbf{f}}(\mathbf{x}) + \widetilde{\mathbf{g}}(\mathbf{x})\mathbf{u}_i - \widetilde{\mathbf{F}}_i = \widetilde{\mathbf{f}}(\mathbf{x}) - \widetilde{\mathbf{f}}(\mathbf{x}_i) + (\widetilde{\mathbf{g}}(\mathbf{x}) - \widetilde{\mathbf{g}}(\mathbf{x}_i))\mathbf{u}_i$$

Uncertainty Sets

Dataset

$$D = \{(\mathbf{x}_i, \mathbf{u}_i, \dot{\mathbf{x}}_i)\}_{i=1}^N$$

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Lipschitz Bounds

$$\left\|\widetilde{\mathbf{f}}(\mathbf{x}) + \widetilde{\mathbf{g}}(\mathbf{x})\mathbf{u}_i - \widetilde{\mathbf{F}}_i\right\| \leq (\mathcal{L}_{\widetilde{\mathbf{f}}} + \mathcal{L}_{\widetilde{\mathbf{g}}}\|\mathbf{u}_i\|_2)\|\mathbf{x} - \mathbf{x}_i\| = \epsilon_i(\mathbf{x})$$

Uncertainty Sets

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Dataset

$$D = \{(\mathbf{x}_i, \mathbf{u}_i, \dot{\mathbf{x}}_i)\}_{i=1}^N$$

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Lipschitz Bounds

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Uncertainty Set

$$\mathcal{U}_i(\mathbf{x}) = \left\{(\mathbf{A}, \mathbf{b}) \in \mathbb{R}^{n\times m} \times \mathbb{R}^n \;\middle|\; \left\|\mathbf{b} + \mathbf{A}\mathbf{u}_i - \widetilde{\mathbf{F}}_i\right\| \leq \epsilon_i(\mathbf{x})\right\} \subset \mathbb{R}^{n\times m} \times \mathbb{R}^n$$

Uncertainty Sets

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Dataset

$$D = \{(\mathbf{x}_i, \mathbf{u}_i, \dot{\mathbf{x}}_i)\}_{i=1}^{N}$$

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$$\widetilde{\mathbf{F}}_i = \dot{\mathbf{x}}_i - (\widehat{\mathbf{f}}(\mathbf{x}_i) + \widehat{\mathbf{g}}(\mathbf{x}_i)\mathbf{u}_i)$$

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Test Point

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Lipschitz Bounds

$$\left\|\widetilde{\mathbf{f}}(\mathbf{x}) + \widetilde{\mathbf{g}}(\mathbf{x})\mathbf{u}_i - \widetilde{\mathbf{F}}_i\right\| \leq (\mathcal{L}_{\widetilde{\mathbf{f}}} + \mathcal{L}_{\widetilde{\mathbf{g}}}\|\mathbf{u}_i\|_2)\|\mathbf{x} - \mathbf{x}_i\| = \epsilon_i(\mathbf{x})$$

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$$(\widetilde{\mathbf{f}}(\mathbf{x}), \widetilde{\mathbf{g}}(\mathbf{x})) \in \mathcal{U}_i(\mathbf{x})$$

Uncertainty Robust Control

Total Uncertainty Set

$$\mathcal{U}(\mathbf{x}) = \cap_{i=1}^{N}\mathcal{U}_i(\mathbf{x}) \subset \mathbb{R}^{n\times m} \times \mathbb{R}^n$$

Uncertainty Robust Control

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Total Uncertainty Set

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Uncertainty Robust Control

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$$(\widetilde{\mathbf{f}}(\mathbf{x}), \widetilde{\mathbf{g}}(\mathbf{x})) \in \mathcal{U}(\mathbf{x})$$

Robust Controller

$$\mathbf{k}_{\text{rob}}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u} - \mathbf{k}_d(\mathbf{x})\|_2^2$$

$$\text{s.t.} \quad \widehat{\dot{\mathtt{C}}}(\mathbf{x},\mathbf{u}) + \nabla\mathtt{C}(\mathbf{x})(\mathbf{b} + \mathbf{A}\mathbf{u}) \leq -\alpha(\mathtt{C}(\mathbf{x}))$$

$$\text{for all } (\mathbf{A}, \mathbf{b}) \in \mathcal{U}(\mathbf{x})$$

Uncertainty Robust Control

Total Uncertainty Set

$$\mathcal{U}(\mathbf{x}) = \cap_{i=1}^{N} \mathcal{U}_i(\mathbf{x}) \subset \mathbb{R}^{n\times m} \times \mathbb{R}^n$$

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Robust Controller

$$\begin{aligned} \mathbf{k}_{\mathrm{rob}}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \ & \|\mathbf{u} - \mathbf{k}_d(\mathbf{x})\|_2^2 \\ \text{s.t.} \ & \widehat{\dot{\mathtt{C}}}(\mathbf{x},\mathbf{u}) + \nabla \mathtt{C}(\mathbf{x})(\mathbf{b} + \mathbf{A}\mathbf{u}) \leq -\alpha(\mathtt{C}(\mathbf{x})) \\ & \text{for all } (\mathbf{A},\mathbf{b}) \in \mathcal{U}(\mathbf{x}) \end{aligned}$$

$$\dot{\mathtt{C}}(\mathbf{x}, \mathbf{k}_{\mathrm{rob}}(\mathbf{x})) \leq -\alpha(\mathtt{C}(\mathbf{x}))$$

Uncertainty Robust Control

Total Uncertainty Set

$$\mathcal{U}(\mathbf{x}) = \cap_{i=1}^{N}\mathcal{U}_i(\mathbf{x}) \subset \mathbb{R}^{n\times m}\times\mathbb{R}^n$$

$$(\widetilde{\mathbf{f}}(\mathbf{x}), \widetilde{\mathbf{g}}(\mathbf{x})) \in \mathcal{U}(\mathbf{x})$$

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$$\text{for all } (\mathbf{A},\mathbf{b}) \in \mathcal{U}(\mathbf{x})$$

$$\dot{\mathtt{C}}(\mathbf{x}, \mathbf{k}_{\mathrm{rob}}(\mathbf{x})) \leq -\alpha(\mathtt{C}(\mathbf{x}))$$

Second-Order Cone Program

$$\mathbf{k}_{\mathrm{rob}}(\mathbf{x}) = \underset{\substack{\mathbf{u}\in\mathbb{R}^m\\ \boldsymbol{\lambda}_i\in\mathbb{R}^n}}{\operatorname{argmin}} \|\mathbf{u}-\mathbf{k}_d(\mathbf{x})\|_2^2$$
$$\text{s.t.}\quad \widehat{\dot{\mathtt{C}}}(\mathbf{x},\mathbf{u}) - \textstyle\sum_{i=1}^{N}\left(\boldsymbol{\lambda}_i^\top\widetilde{\mathbf{F}}_i - \|\boldsymbol{\lambda}_i\|_2\epsilon_i(\mathbf{x})\right) \leq -\alpha(\mathtt{C}(\mathbf{x}))$$
$$\textstyle\sum_{i=1}^{N}\boldsymbol{\lambda}_i\mathbf{u}_i^\top = -\nabla\mathtt{C}(\mathbf{x})^\top\mathbf{u}^\top$$
$$\textstyle\sum_{i=1}^{N}\boldsymbol{\lambda}_i = -\nabla\mathtt{C}(\mathbf{x})^\top$$

Uncertainty Visualization

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Control Certificate Function Derivative Uncertainty Set

$$\widetilde{\mathcal{U}}_{\mathtt{C}}(\mathbf{x}) = \left\{ (\mathbf{a}, b) \in \mathbb{R}^m \times \mathbb{R} \;\middle|\; \exists (\mathbf{A}, \mathbf{b}) \in \mathcal{U}(\mathbf{x}) \text{ s.t. } \mathbf{a} = (\nabla \mathtt{C}(\mathbf{x})\mathbf{A})^\top, b = \nabla \mathtt{C}(\mathbf{x})\mathbf{b} \right\}$$

Uncertainty Visualization

Control Certificate Function Derivative Uncertainty Set

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↓

Control Certificate Function Derivative Set

$$\mathcal{U}_{\mathrm{C}}(\mathbf{x}) = \left\{\left(\nabla \mathrm{C}(\mathbf{x})\widehat{\mathbf{g}}(\mathbf{x}), \nabla \mathrm{C}(\mathbf{x})\widehat{\mathbf{f}}(\mathbf{x})\right)\right\} \oplus \widetilde{\mathcal{U}}_{\mathrm{C}}(\mathbf{x})$$

Uncertainty Visualization

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Control Certificate Function Derivative Set

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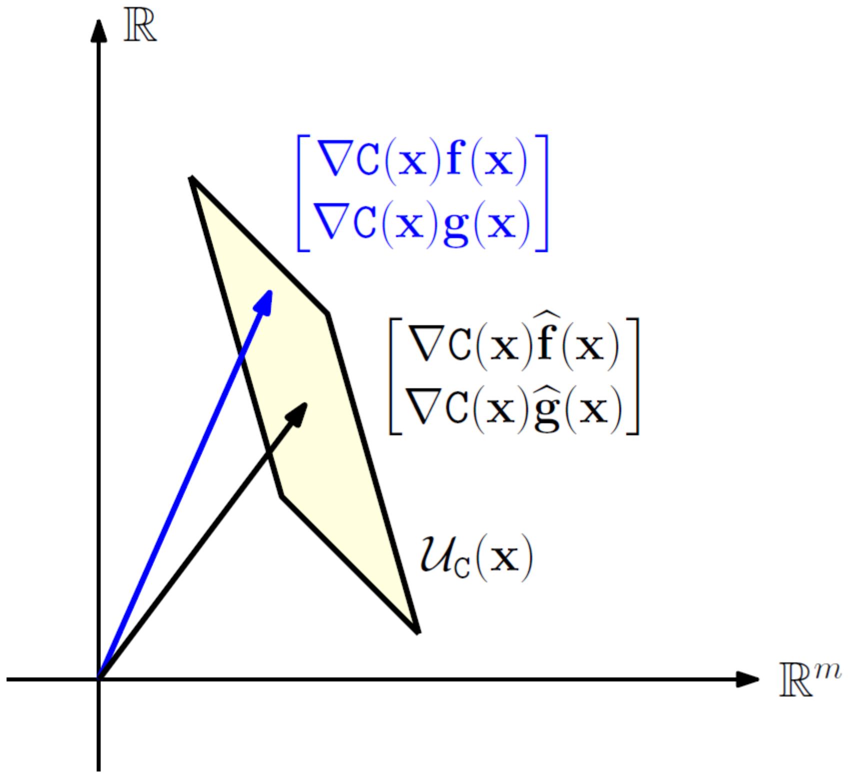

Uncertainty Visualization

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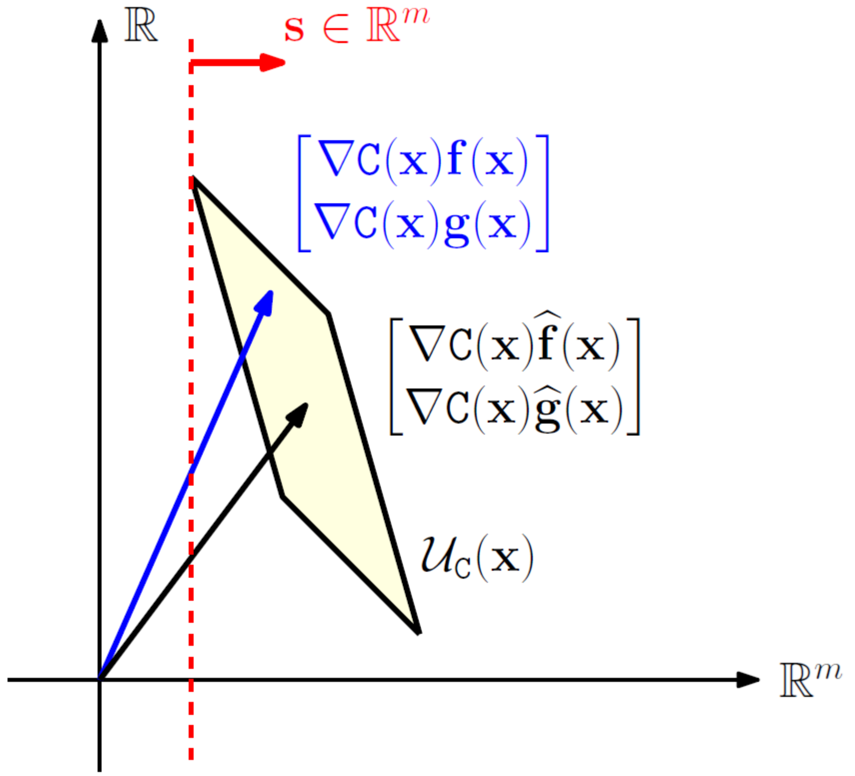
Separating Hyperplane

$$\langle \mathbf{s}, \mathbf{a} \rangle > \alpha > 0 \quad \forall (\mathbf{a}, b) \in \mathcal{U}_{\mathsf{C}}(\mathbf{x})$$



Uncertainty Visualization

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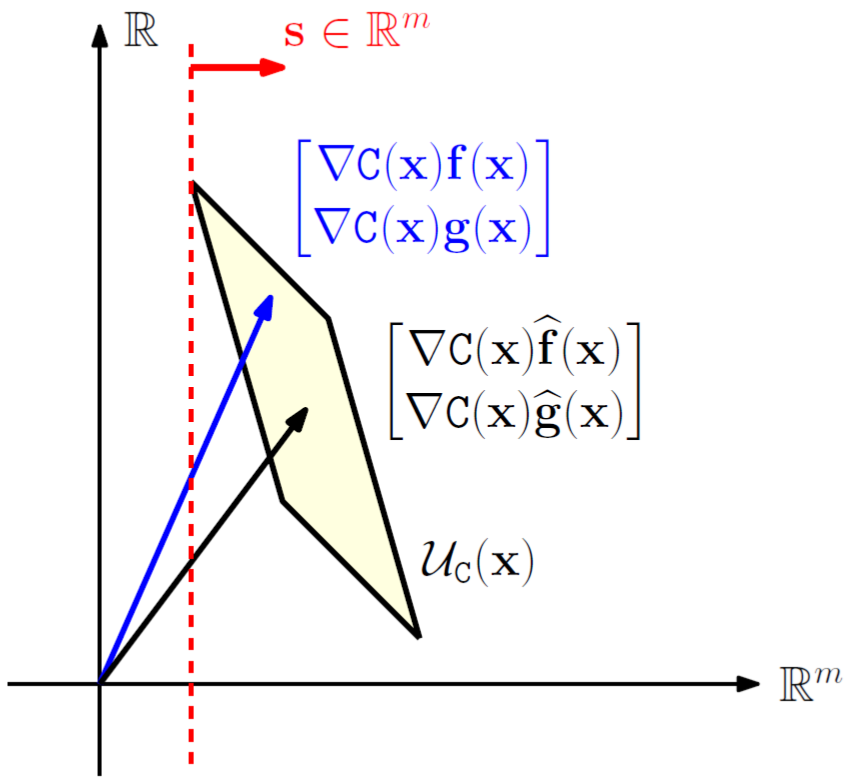
Choose inputs along s!



Optimization Feasibility

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Compact Uncertainty Sets

Lemma 1 (Bounded Uncertainty Sets). *Consider a dataset D with N data points satisfying $N \geq m+1$. If there exists a set of data points $\{(\mathbf{x}_i, \mathbf{u}_i, \dot{\mathbf{x}}_i)\}_{i=1}^{m+1} \subseteq D$ such that the set of vectors:*

$$\mathcal{M} \triangleq \left\{ \begin{bmatrix} \mathbf{u}_i^\top & 1 \end{bmatrix}^\top \right\}_{i=1}^{m+1}, \tag{16}$$

are linearly independent, then the uncertainty set $\mathcal{U}(\mathbf{x})$ is bounded (and thus compact) for any $\mathbf{x} \in \mathbb{R}^n$.

Optimization Feasibility

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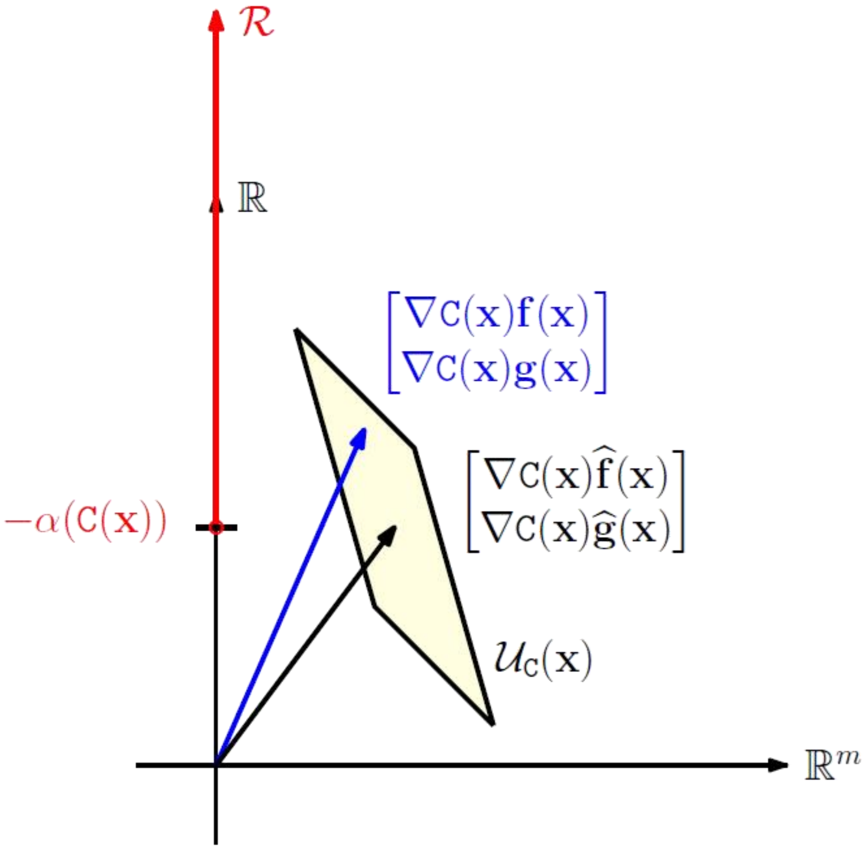
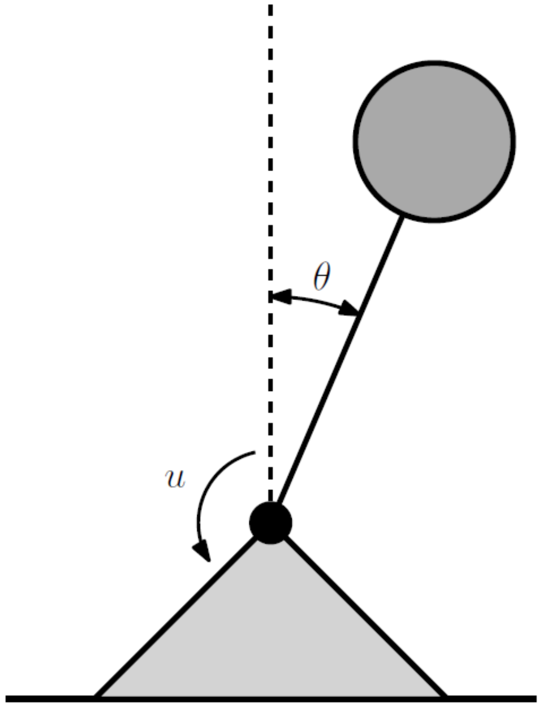
CCF Condition Under Uncertainty

Theorem 2 (Feasibility of Data-Driven Robust Controller). *For a state $\mathbf{x} \in \mathbb{R}^n$, define the ray $\mathcal{R} \subset \mathbb{R}^{m+1}$ as $\mathcal{R} = \{\mathbf{0}_m\} \times (-\alpha(\mathtt{C}(\mathbf{x})), \infty)$. Assuming that $\mathcal{U}(\mathbf{x})$ is bounded, the data-driven robust controller is feasible if and only if:*

$$\mathcal{U}_{\mathtt{C}}(\mathbf{x}) \cap \mathcal{R} = \emptyset. \tag{20}$$



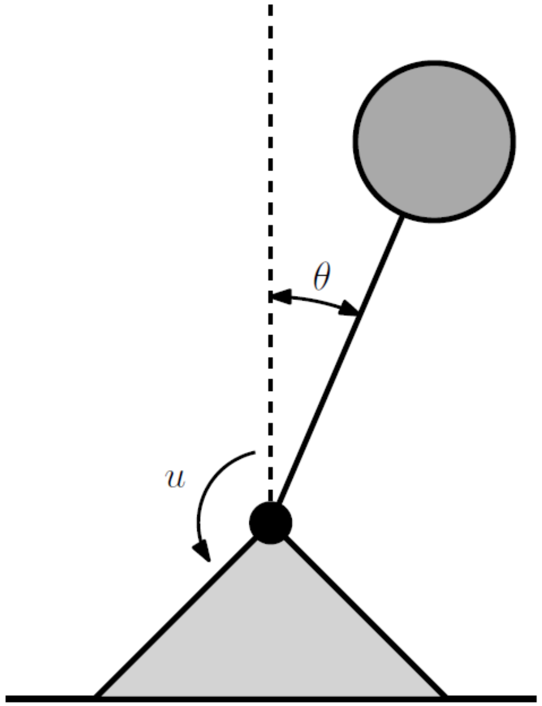
Inverted Pendulum



Model

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ \frac{\widehat{g}}{l}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0 \\ \frac{1}{\widehat{m}\widehat{l}^2}\end{bmatrix} u$$

Inverted Pendulum



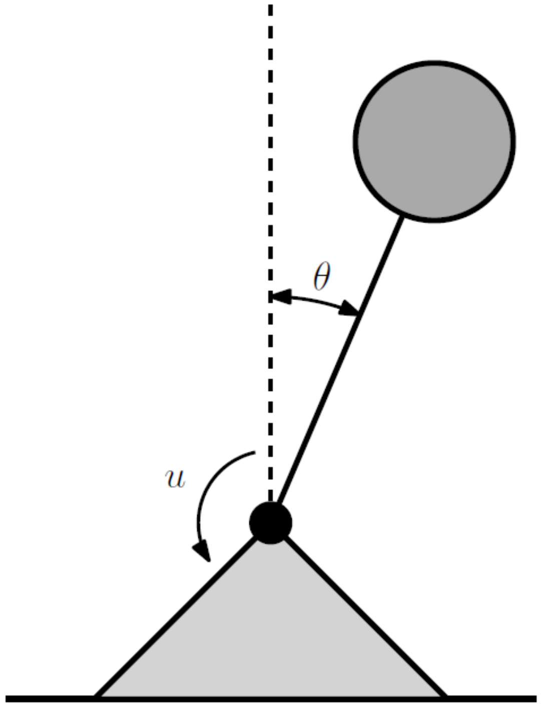
Model

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{\widehat{g}}{\widehat{l}}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{\widehat{m}\widehat{l}^2}\end{bmatrix}u$$

True System

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{g}{l}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1-0.75e^{-\theta^2}}{ml^2}\end{bmatrix}u$$

Inverted Pendulum



Model

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{\hat{g}}{\hat{l}}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{\widehat{ml^2}}\end{bmatrix}u$$

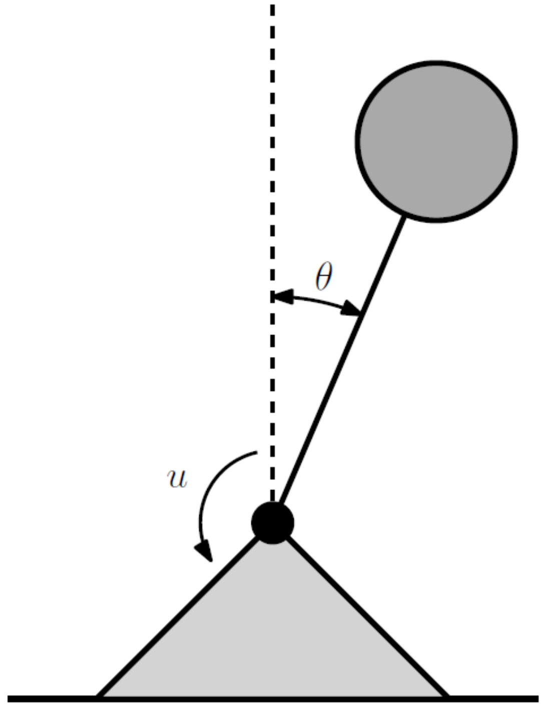
True System

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{g}{l}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1-0.75e^{-\theta^2}}{ml^2}\end{bmatrix}u$$

State Data

$$(\theta_i, \dot{\theta}_i) \in \{0, 0.025, \dots, 1\} \times \{-0.25, -0.225, \dots, 0.25\}$$

Inverted Pendulum



Model

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{g}{\hat{l}}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{\widehat{m}\widehat{l}^2}\end{bmatrix}u$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{g}{l}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1-0.75e^{-\theta^2}}{ml^2}\end{bmatrix}u$$

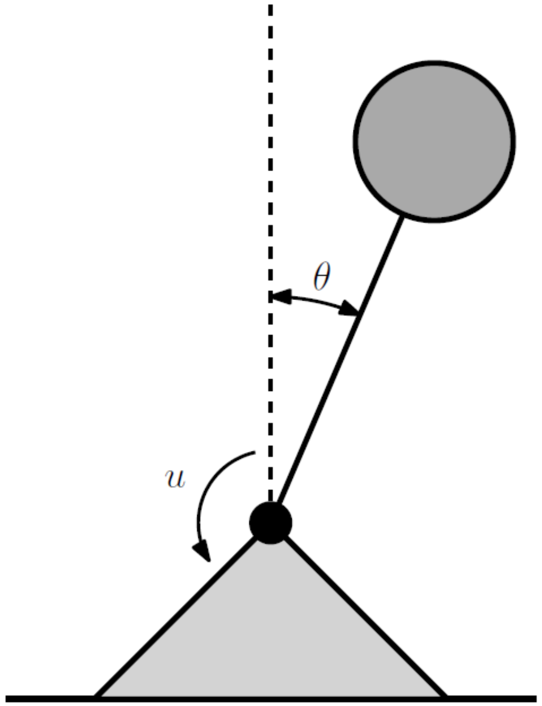
State Data

$$(\theta_i, \dot{\theta}_i) \in \{0, 0.025, \dots, 1\} \times \{-0.25, -0.225, \dots, 0.25\}$$

Dense Input Data

$$\text{Dense} : u_i \in \{-5, -3, -1, 1, 3, 5\}$$

Inverted Pendulum



Model

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{g}{\hat{l}}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{\widehat{m}\widehat{l}^2}\end{bmatrix}u$$

True System

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \frac{g}{l}\sin(\theta)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1-0.75e^{-\theta^2}}{ml^2}\end{bmatrix}u$$

State Data

$$(\theta_i, \dot{\theta}_i) \in \{0, 0.025, \dots, 1\} \times \{-0.25, -0.225, \dots, 0.25\}$$

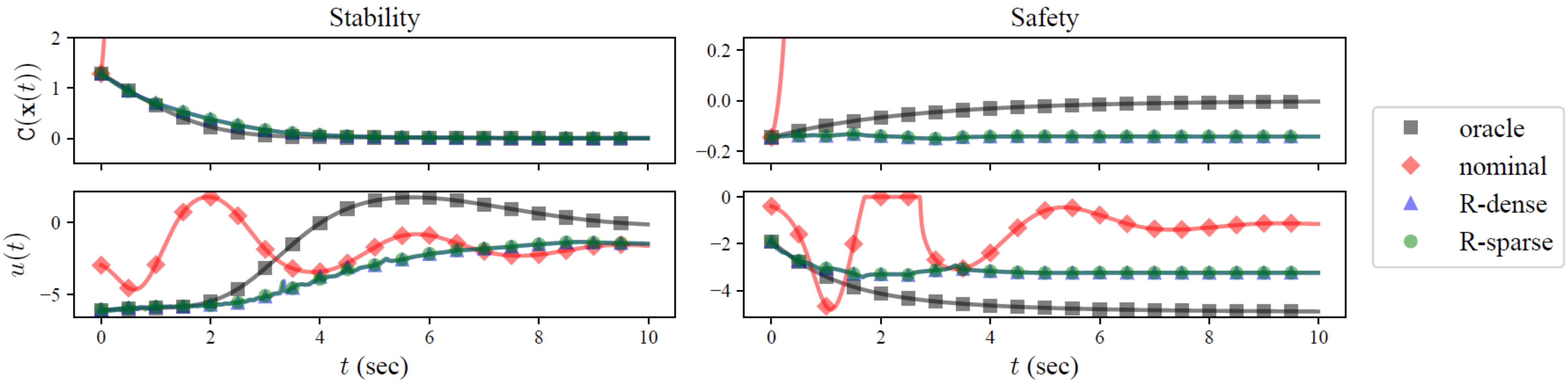
Dense Input Data

$$\text{Dense} : u_i \in \{-5, -3, -1, 1, 3, 5\}$$

Sparse Input Data

$$\text{Sparse} : u_i \in \{-5, -1\}$$

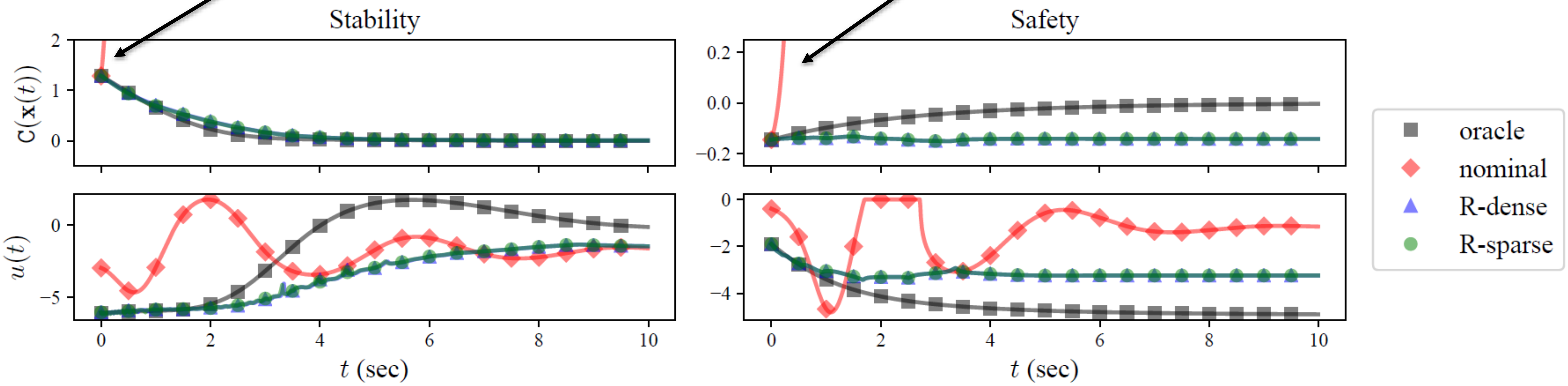
Inverted Pendulum



Inverted Pendulum

Nominal controller unstable!

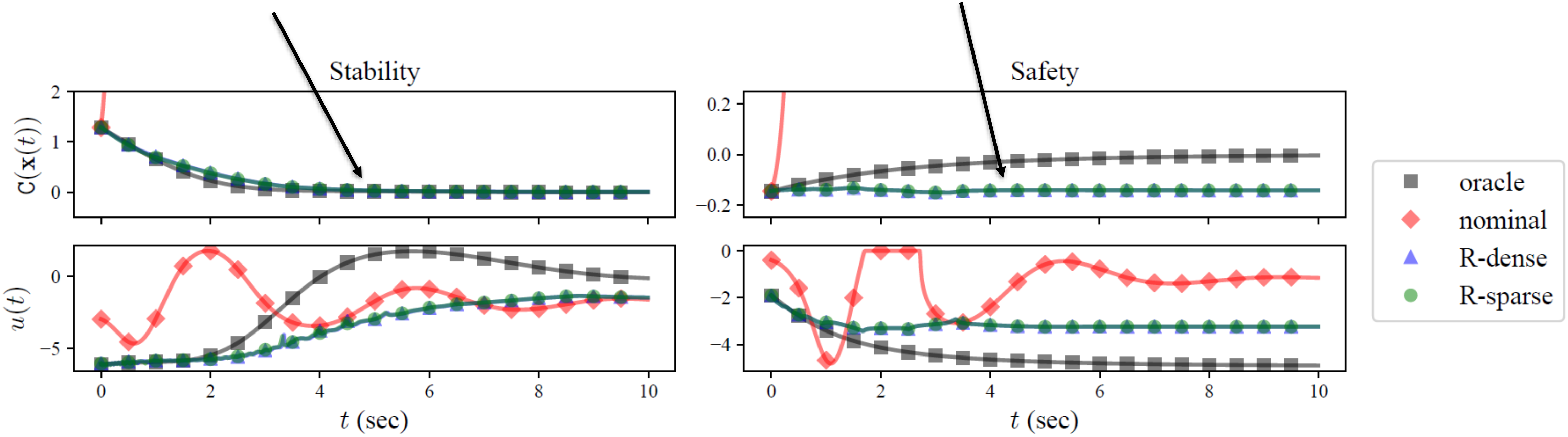
Nominal controller unsafe!



Inverted Pendulum

Data-driven controllers stable!

Data-driven controllers safe!



Inverted Pendulum

Data-driven controllers stable!

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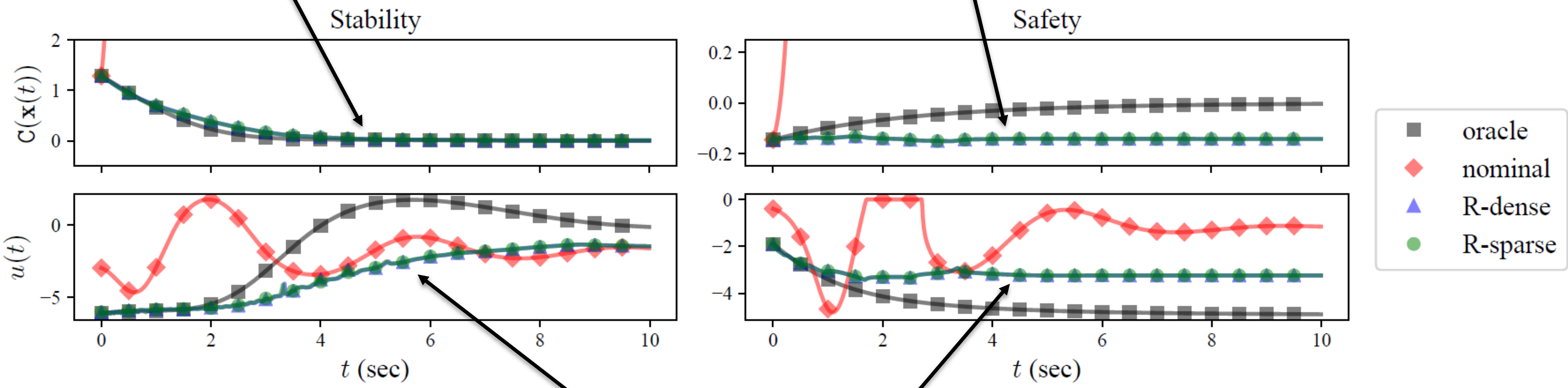
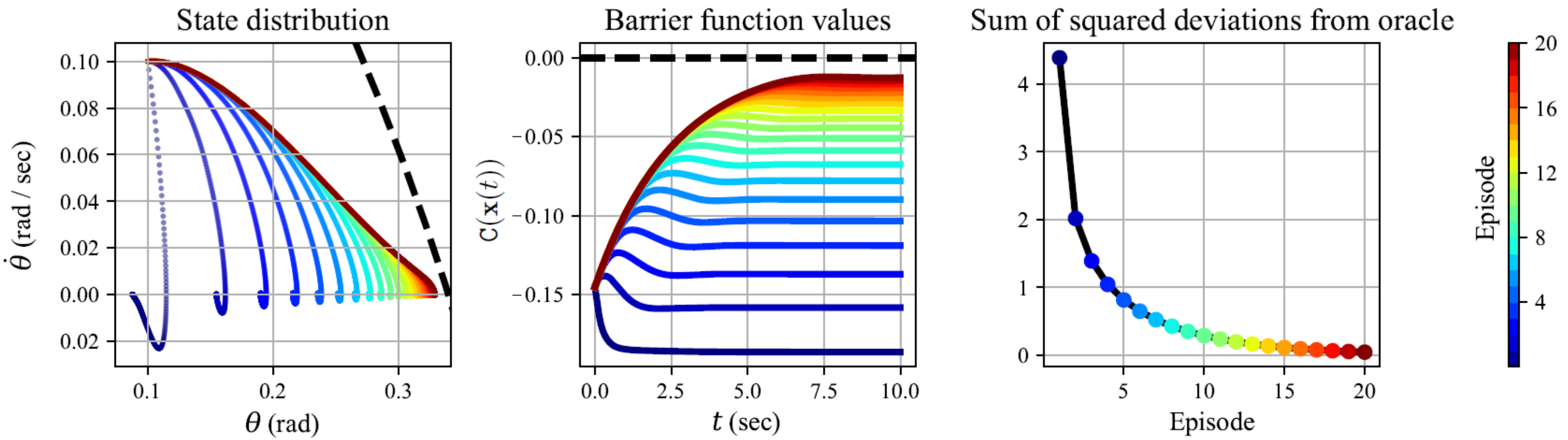


Sparse and dense controllers perform nearly the same!

Episodic Learning



Conclusions

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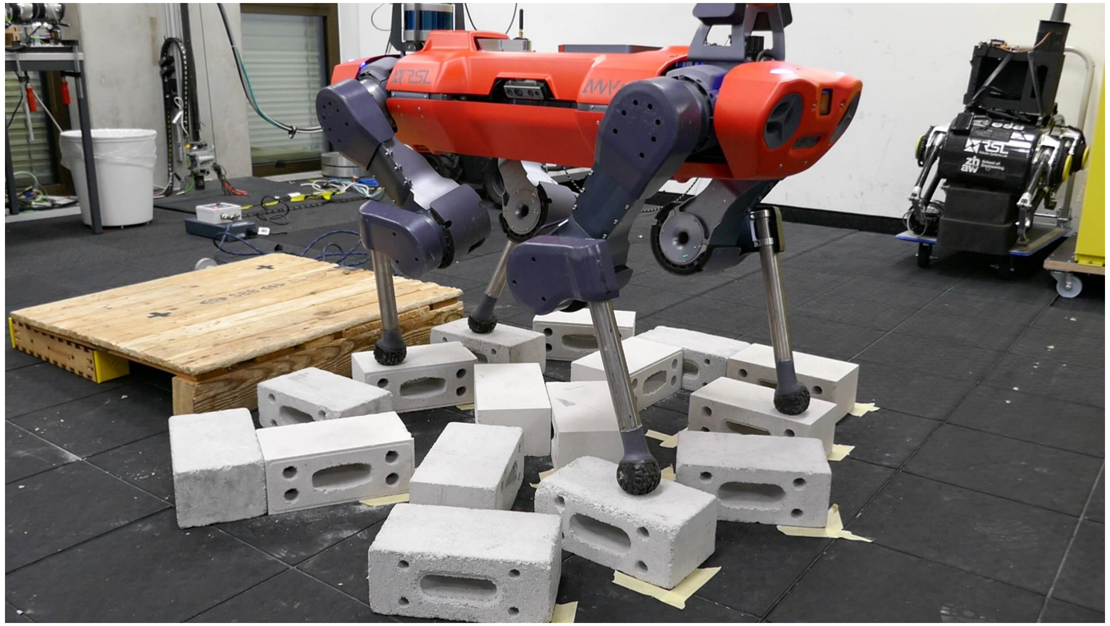
- **Actuation uncertainty** plays a large role in robust control design

- **Robust convex optimization** enables the design of controllers even under partial characterization of actuation

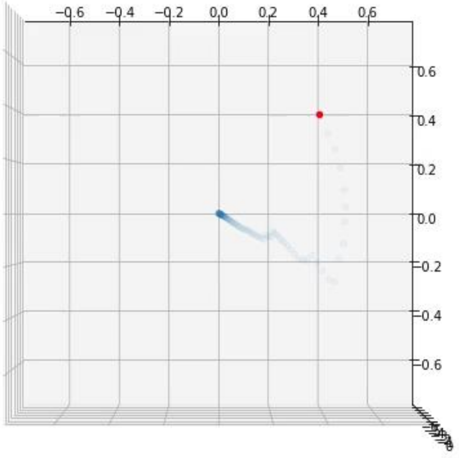
- **Data-driven** control design via CCFs enables a theoretical understanding of controller properties in terms of data



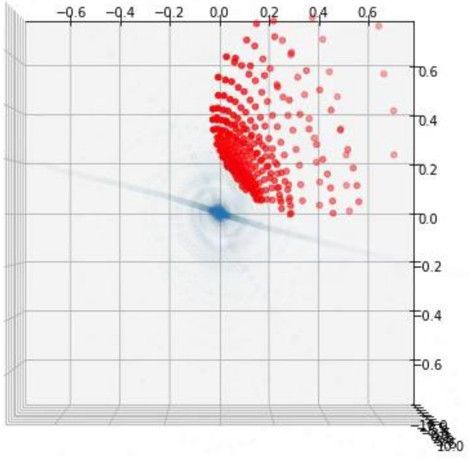
Next Steps

- **Efficient** down-selection of data for use in optimal controller
- **Sampled-data** control design to support sample-hold inputs
- **Exploration** schemes for collecting input data

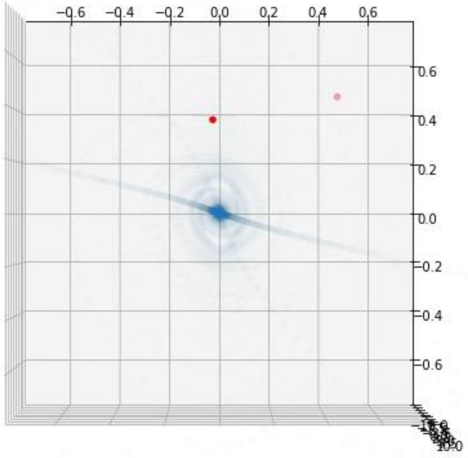
State Trajectory



Diverse Data (N~1e3)



Controller Data (N~1e0)



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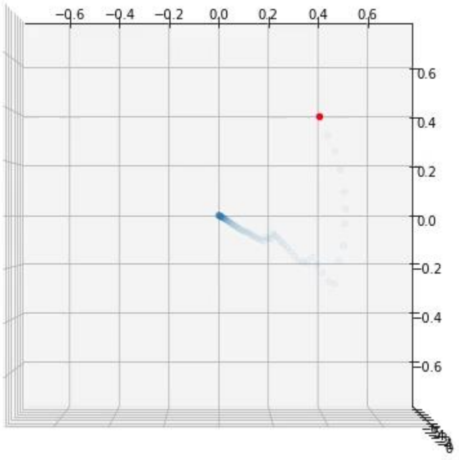
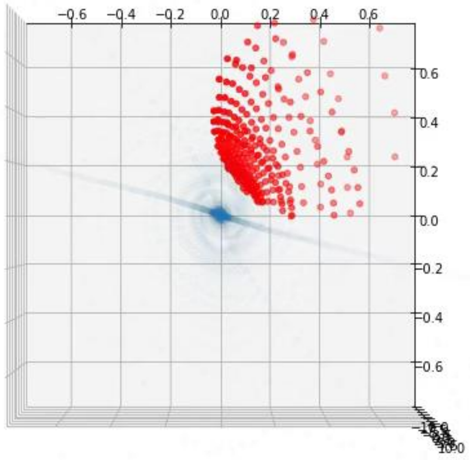
[18] F. Castañeda, J. J. Choi, B. Zhang, C. J. Tomlin, K. Sreenath, "Gaussian Process-based Min-norm Stabilizing Controller for Control-Affine Systems with Uncertain Input Effects", 2020.
[19] F. Castañeda, J. J. Choi, B. Zhang, C. J. Tomlin, K. Sreenath, "Pointwise Feasibility of Gaussian Process-based Safety-Critical Control under Model Uncertainty", 2021.

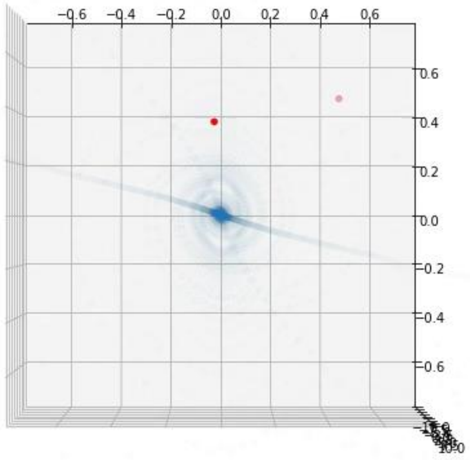
State Trajectory



Diverse Data (N~1e3)



Controller Data (N~1e0)



Thank You!

Towards Robust Data-Driven Control Synthesis for Nonlinear Systems with Actuation Uncertainty

Andrew Taylor Victor Dorobantu Sarah Dean
Benjamin Recht Yisong Yue Aaron Ames