Sampled-Data Stabilization with Control Lyapunov Functions via Quadratically Constrained Quadratic Programs

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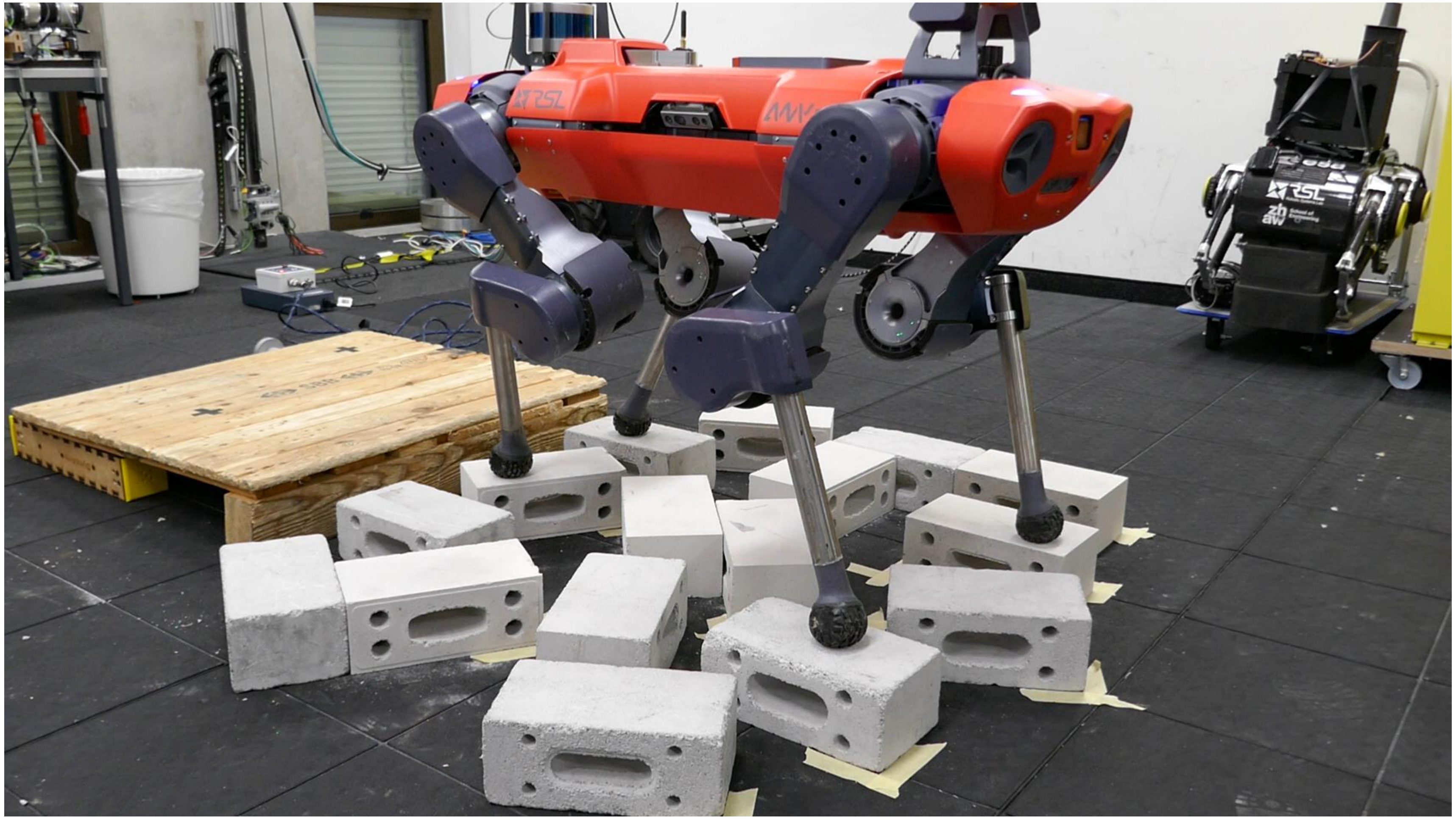
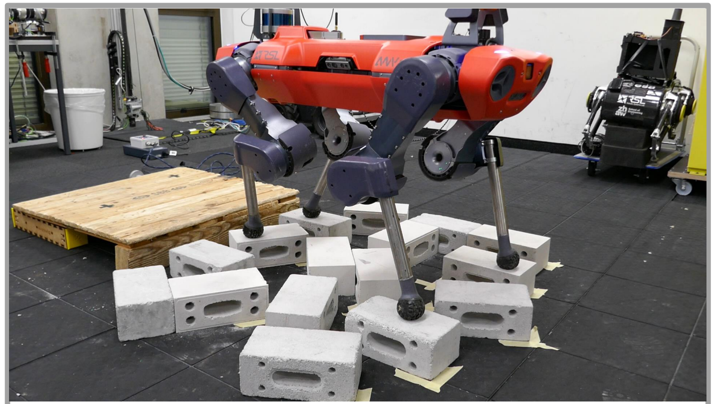
Control in the real world is hard

But: Pretty when it works…



[1] R. Grandia, **A. J. Taylor**, M. Hutter, A. D. Ames, "Multi-Layered Safety for Legged Robotics via Control Barrier Functions and Model Predictive Control", 2020.

Claim: Need to Bridge the Gap



$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2$$
$$\text{s.t. } \dot{V}(\mathbf{x},\mathbf{u}) \leq -\alpha(V(\mathbf{x}))$$

Bridge the Gap

$$V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) \leq -\alpha(V(\mathbf{x}_k))$$

Theorems & Proofs

Experimental Realization

Contributions

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- Framework for achieving sampled-data control for system stability via **Control Lyapunov Functions (CLFs)**
- Constructive process for synthesizing discrete CLFs using feedback linearizability and approximate transition maps
- Analysis of stability of zero-dynamics for sampled-data systems

System Dynamics

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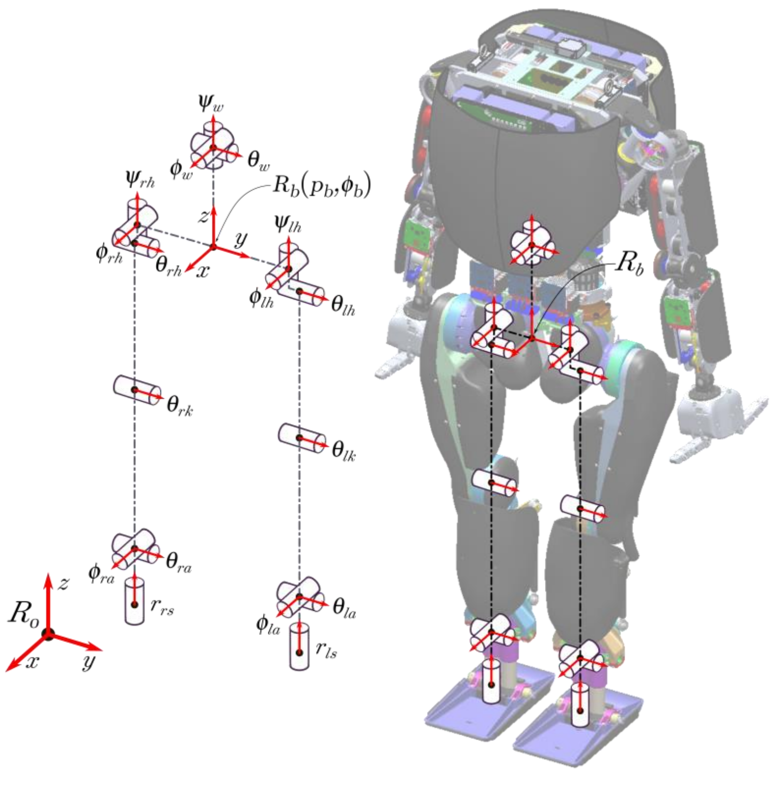
Equations of Motion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \qquad \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^{n \times m}$$



Mathematical Model **System Model**

System Dynamics

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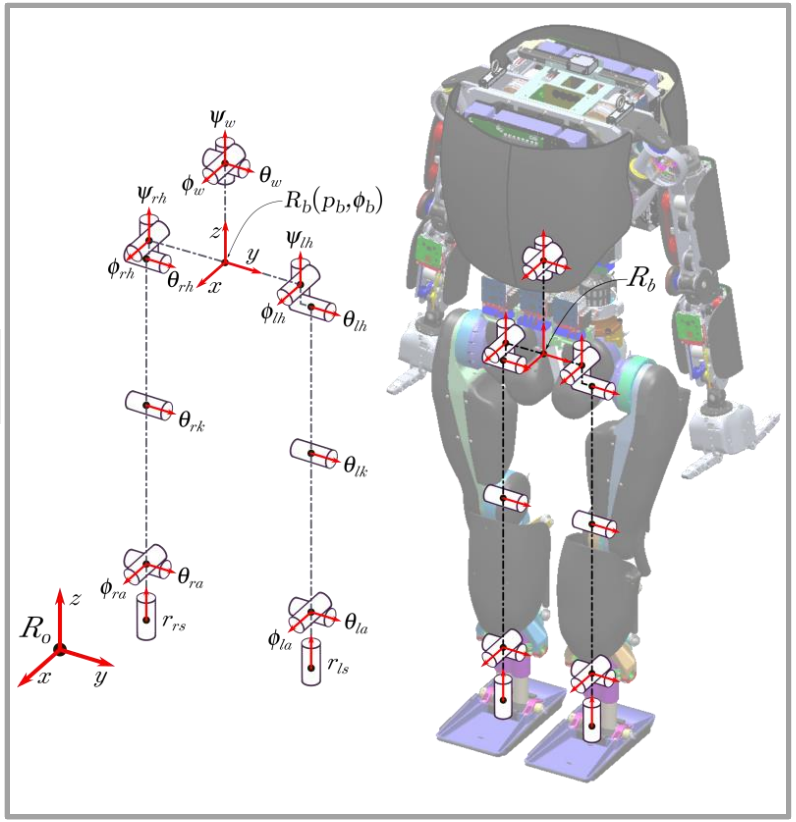
Assumptions

$\mathbf{f}, \ \mathbf{g}$ locally Lipschitz continuous

$$\mathbf{f}(\mathbf{0}) = \mathbf{0}$$

Mathematical Model



System Model

Feedback Linearization

Caltech UCLA

State Dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

Feedback Linearization

State Dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \boldsymbol{\Phi}(\mathbf{x})$$

$$\boldsymbol{\Phi}(\mathbf{0}) = \mathbf{0}$$

Normal Dynamics

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\mathbf{z}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix}}_{\mathbf{f}_{\boldsymbol{\xi}}} + \underbrace{\begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{0} \end{bmatrix}}_{\mathbf{g}_{\boldsymbol{\xi}}} \mathbf{u}$$

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$$\mathbf{u} = \mathbf{k}_{\mathrm{fbl}}(\boldsymbol{\xi})$$

Linearized Dynamics

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Zero Dynamics

$$\dot{\mathbf{z}} = \mathbf{q}(\mathbf{0}, \mathbf{z})$$

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Exponential Stability

$$\dot{\boldsymbol{\xi}} = \mathbf{f}_{\boldsymbol{\xi}}(\boldsymbol{\xi}) + \mathbf{g}_{\boldsymbol{\xi}}(\boldsymbol{\xi})\mathbf{k}_{\mathrm{fbl}}(\boldsymbol{\xi})$$

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Control Lyapunov Functions (CLFs)

Lyapunov Equation

$$\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

$$\mathbf{Q} \succ 0 \quad \mathbf{P} \succ 0$$

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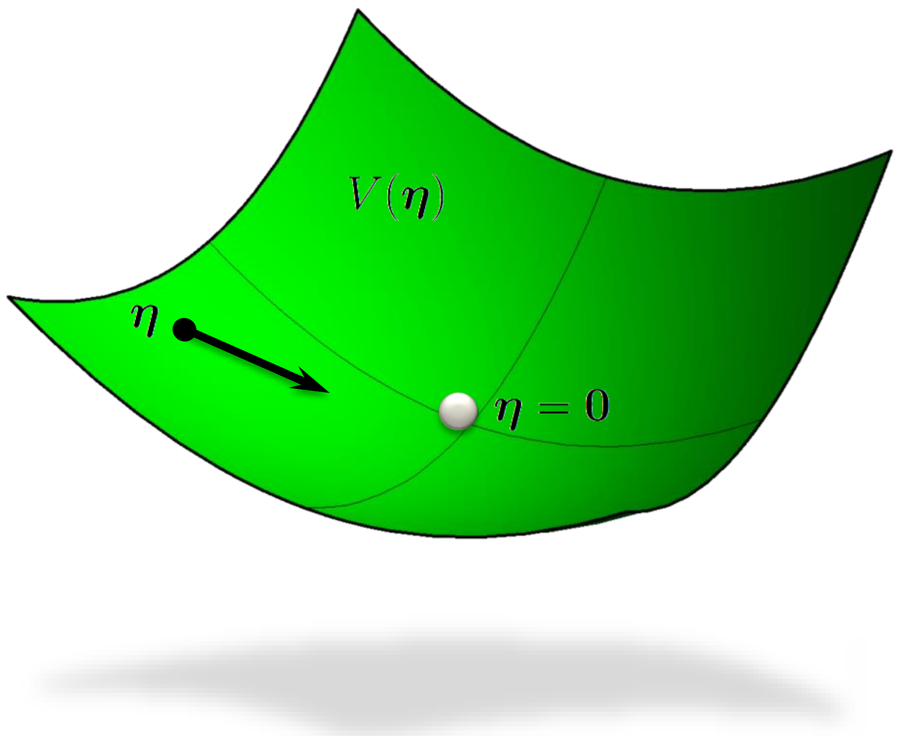

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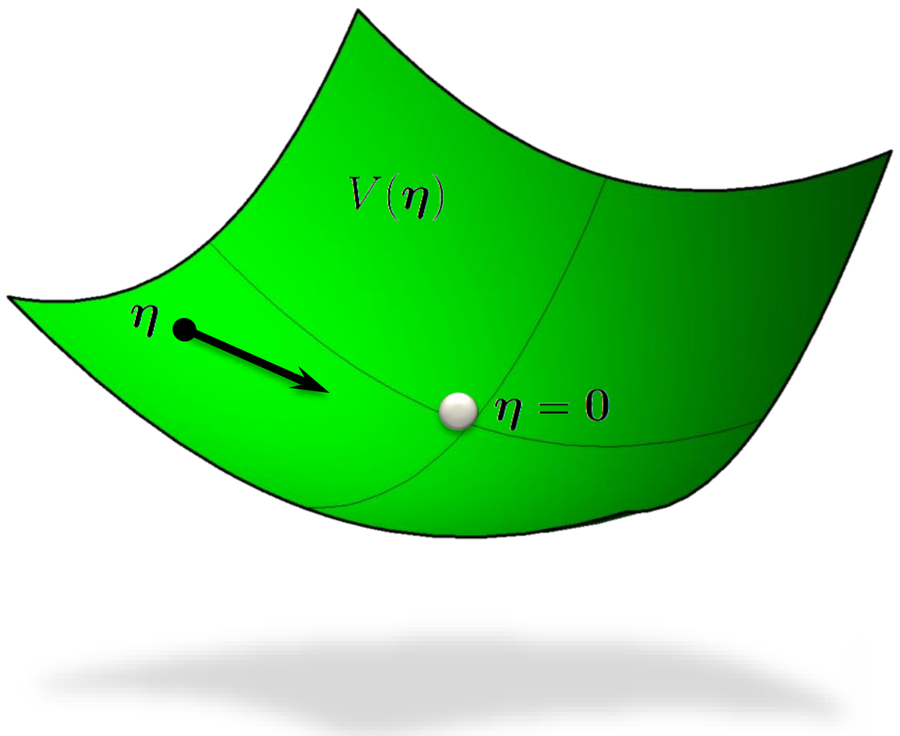

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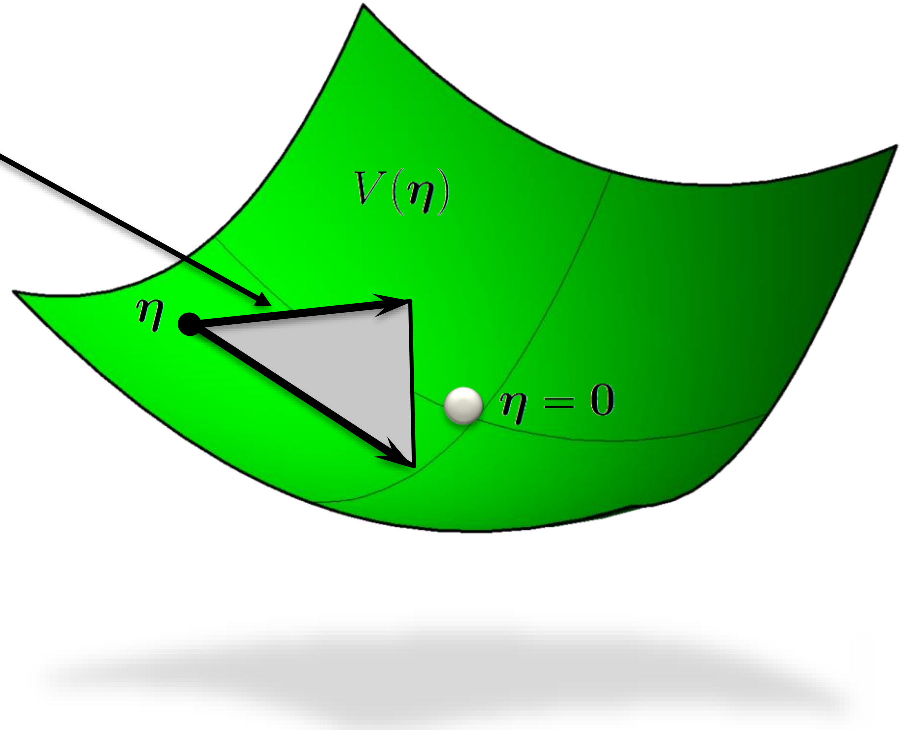

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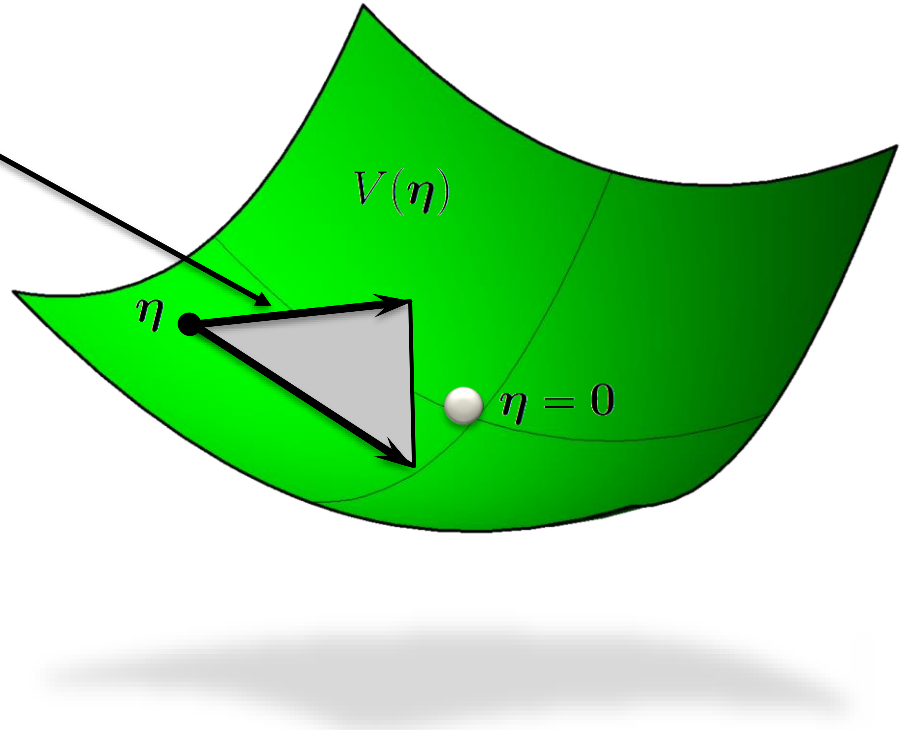
Feedback Controllers

[2] Z. Artstein, "Stabilization with relaxed controls", 1983.
[3] E. Sontag, "A universal construction of Artstein's theorem on nonlinear stabilization", 1989.
[4] R. Freeman, P. Kokotovic, "Inverse Optimality in Robust Stabilization", 1996.



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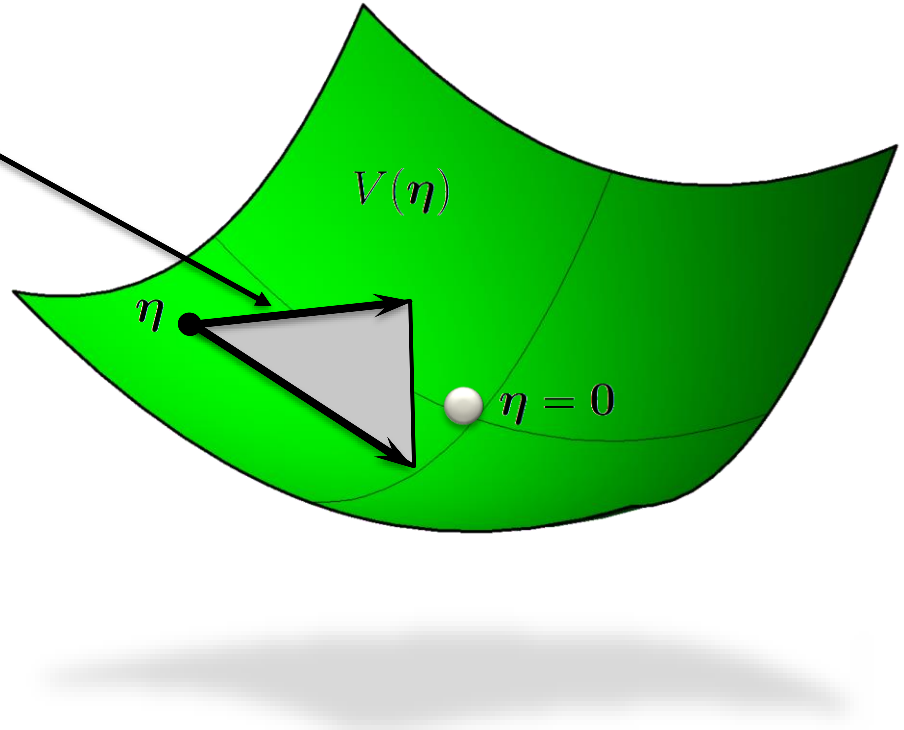

[5] A. Ames, M. Powell, "Towards the unification of locomotion and manipulation through control lyapunov functions and quadratic programs", 2013.

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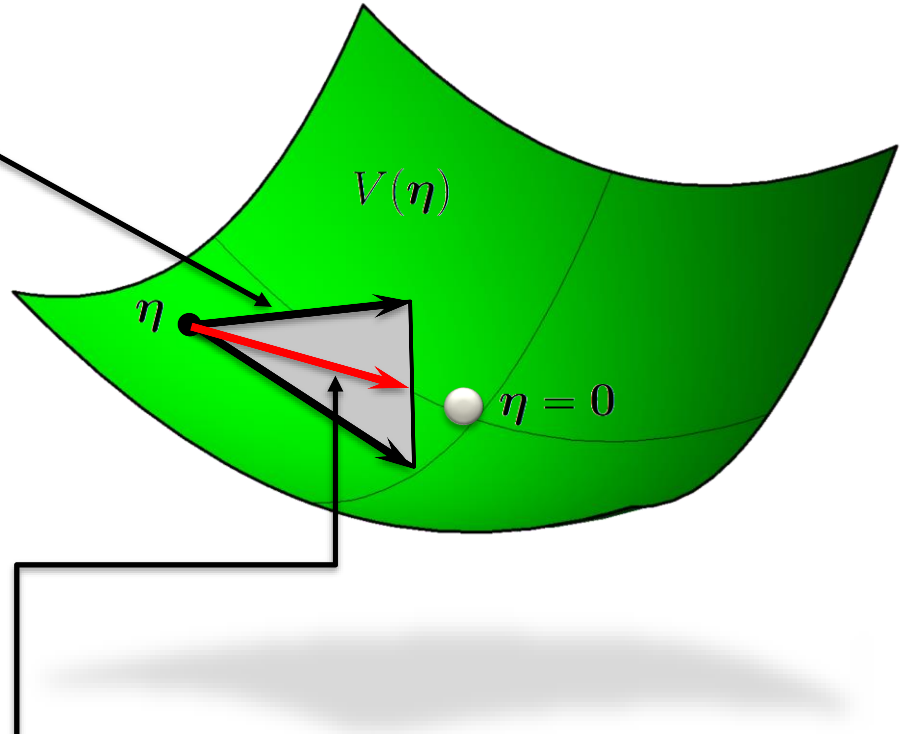

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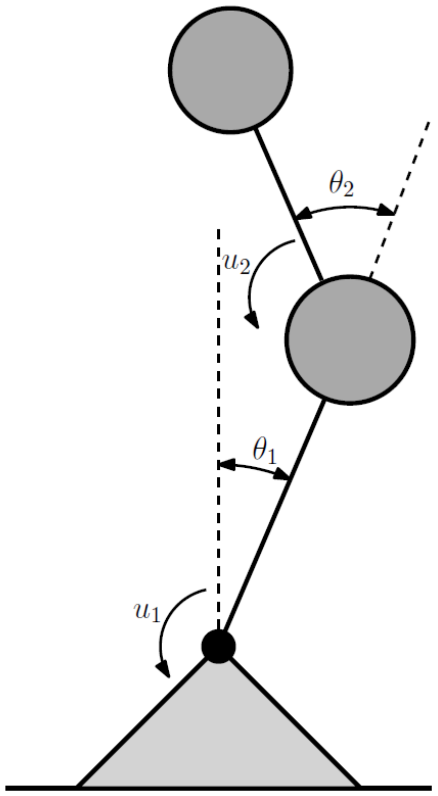

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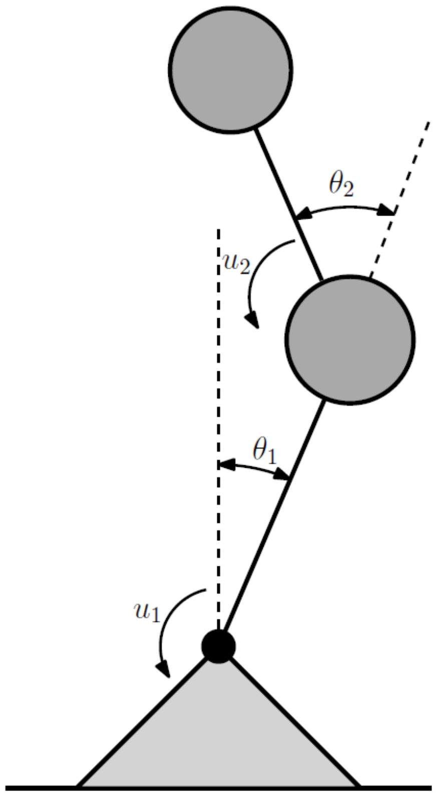
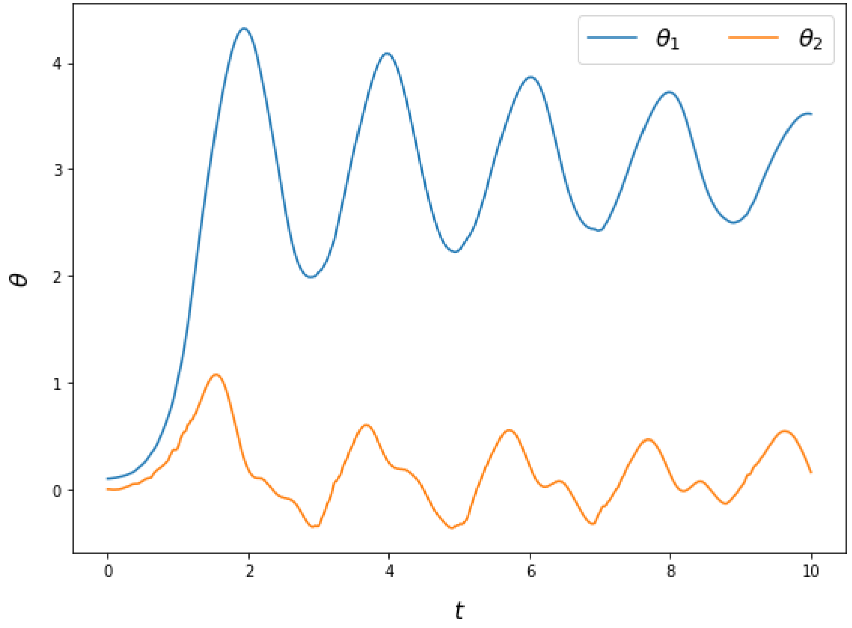
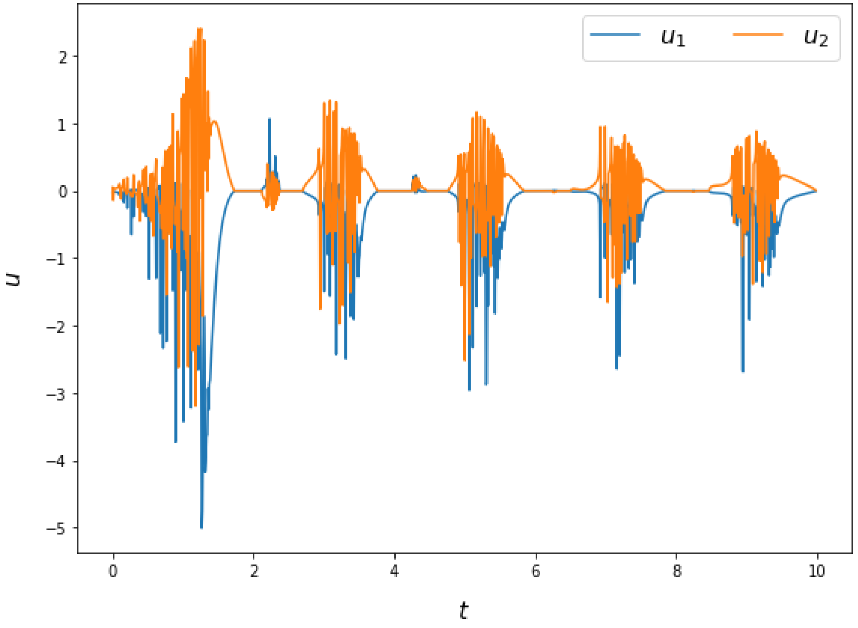
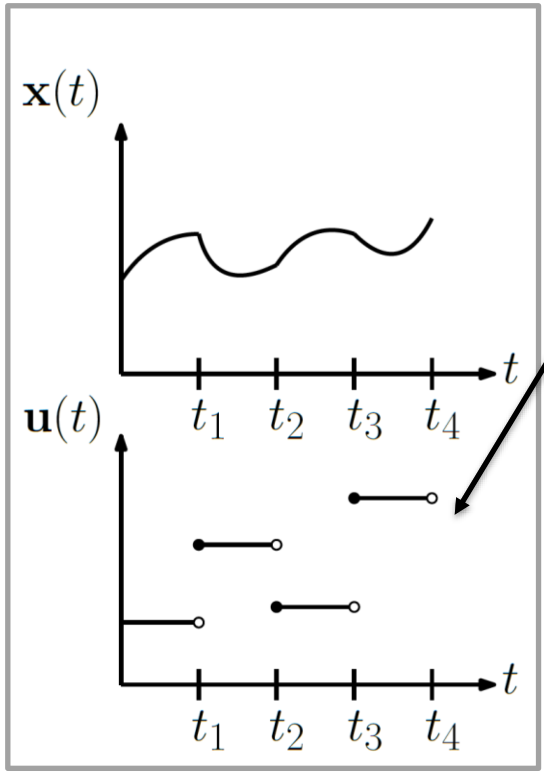






Sampled-Data Control



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \ \forall t \in [t_i, t_{i+1})$$

$$t_{i+1} - t_i = h$$

$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + \int_{t_i}^{t_{i+1}} \mathbf{f}(\mathbf{x}(\tau)) + \mathbf{g}(\mathbf{x}(\tau))\mathbf{k}(\mathbf{x}(t_i)) \, \mathrm{d}\tau$$

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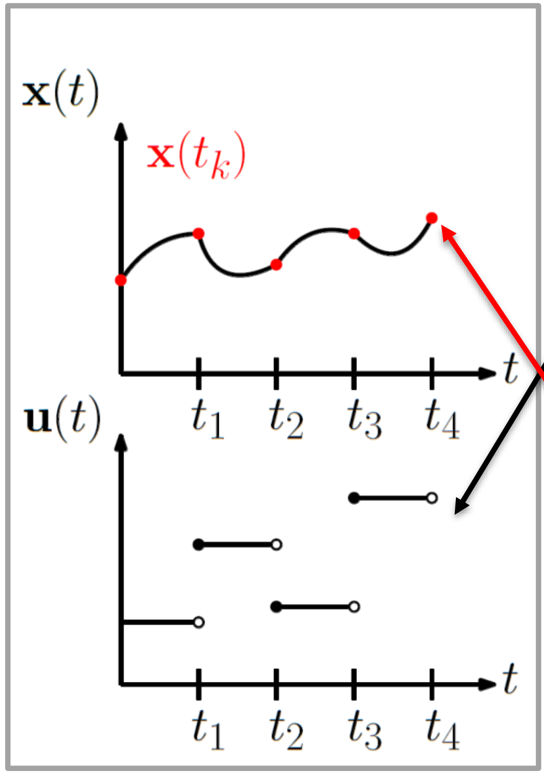

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Discrete State Transitions

$$\mathbf{F}_h^e(\mathbf{x}, \mathbf{u}) = \mathbf{x} + \int_0^h \mathbf{f}(\mathbf{x}(\tau)) + \mathbf{g}(\mathbf{x}(\tau))\mathbf{u} \ \mathrm{d}\tau$$

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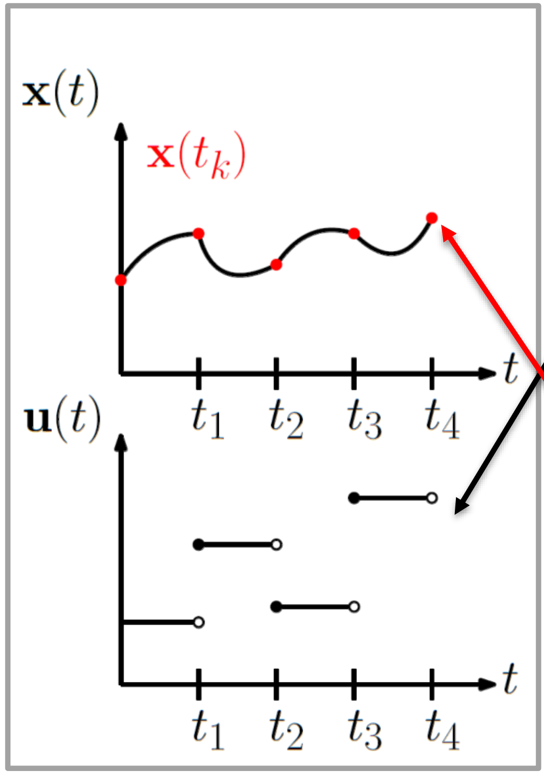

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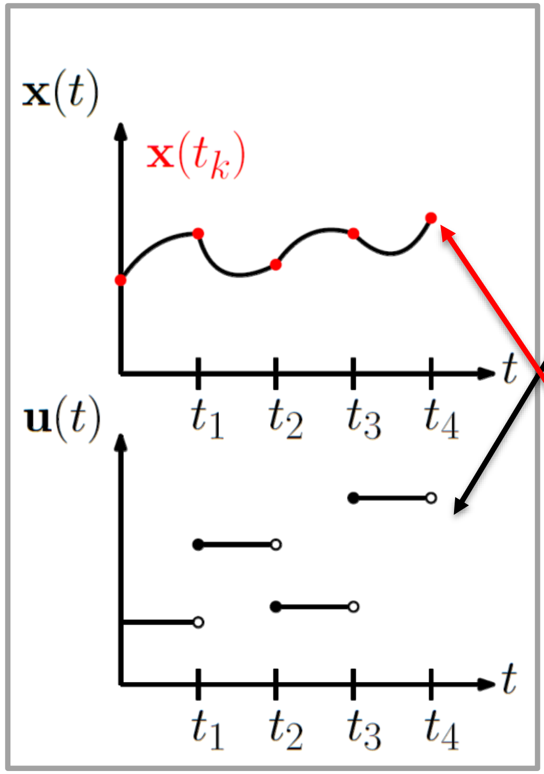
How do we work with?

Sampled-Data Control



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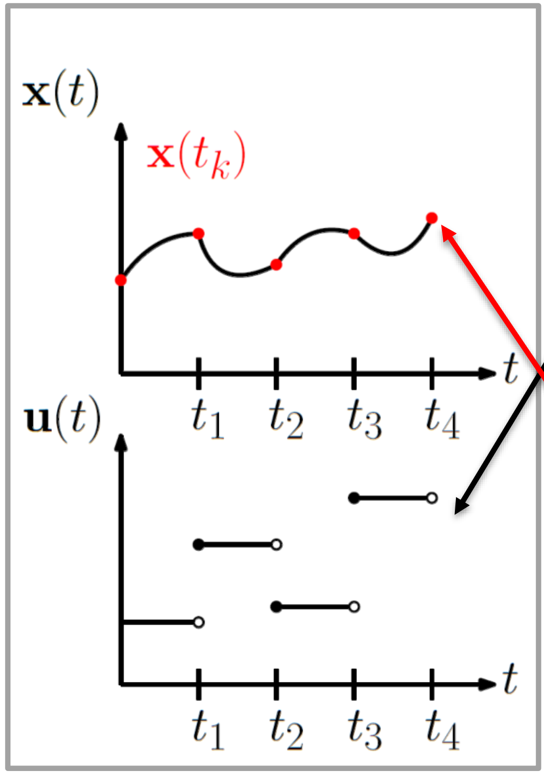
[6] S. Monaco, D. Normand-Cyrot, "Advanced Tools for Nonlinear Sampled-Data Systems Analysis and Control", 2007.

Sampled-Data Control



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How do we work with?

Approximate

Approximate Discrete Models

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$$\mathbf{F}_h^a(\mathbf{x},\mathbf{u}) \approx \mathbf{F}_h^e(\mathbf{x},\mathbf{u})$$

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Stability Analysis

[7] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999
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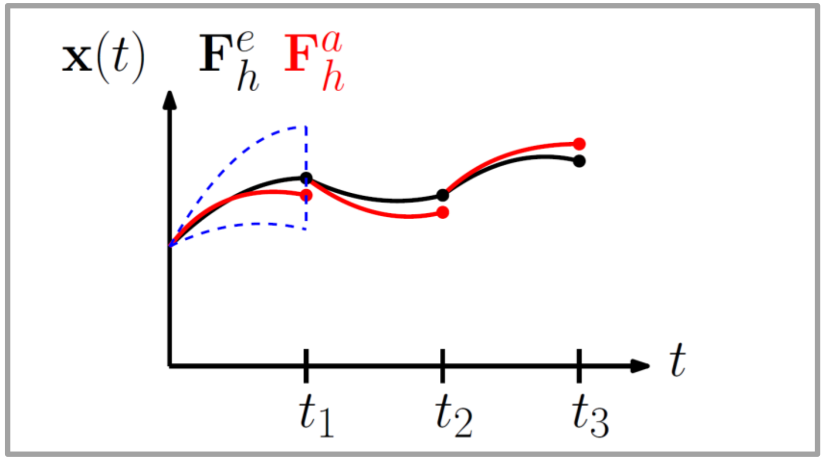
One-Step Consistency

$$\|\mathbf{F}_h^a(\mathbf{x},\mathbf{u}) - \mathbf{F}_h^e(\mathbf{x},\mathbf{u})\| \leq h\rho(h)$$
$$\rho \in \mathcal{K} \quad h \in (0, h_{\max}]$$



Approximate Discrete Models

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Approximate Discrete Model

$$\mathbf{F}_h^a(\mathbf{x},\mathbf{u}) \approx \mathbf{F}_h^e(\mathbf{x},\mathbf{u})$$

Stability Analysis

[7] D. Nešić, A. Teel, P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", 1999
[8] D. Nešić, A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models", 2004

One-Step Consistency

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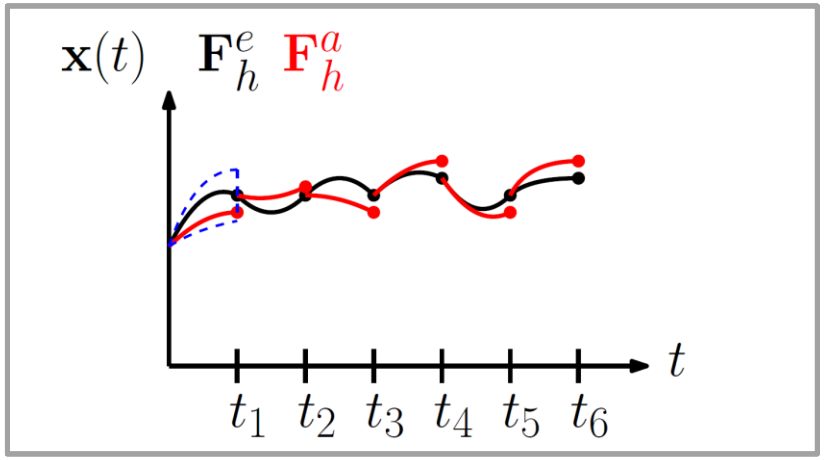

Approximate Discrete Models

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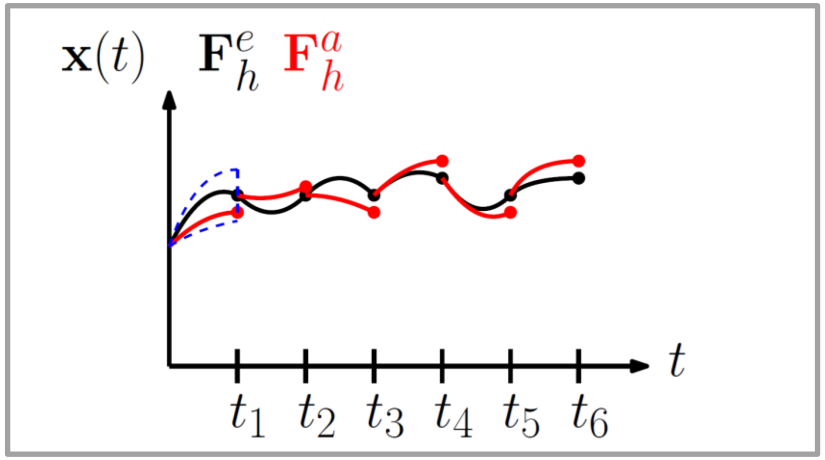

Euler Approximate Model

$$\mathbf{F}^a_h(\mathbf{x},\mathbf{u}) = \mathbf{x} + h(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u})$$

Approximate Discrete Models

Approximate Discrete Model

$$\mathbf{F}^a_h(\mathbf{x},\mathbf{u}) \approx \mathbf{F}^e_h(\mathbf{x},\mathbf{u})$$

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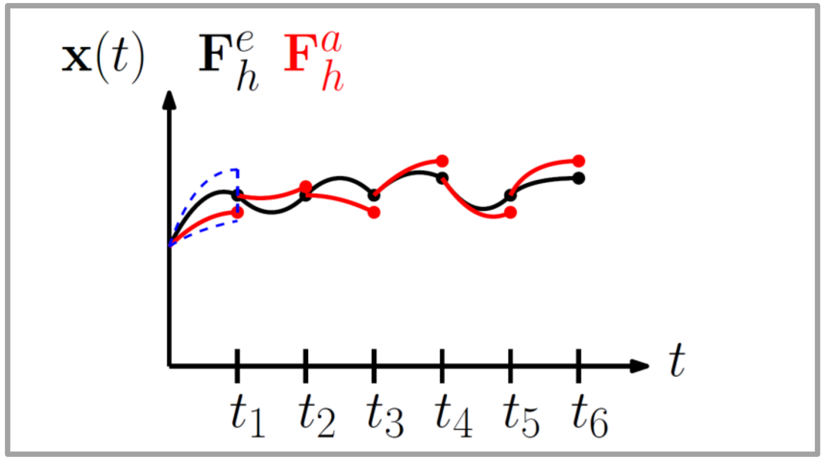

Euler Approximate Model

$$\mathbf{F}^a_h(\mathbf{x},\mathbf{u}) = \mathbf{x} + h(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u})$$

$\mathbf{f},\ \mathbf{g}$ locally Lipschitz $\implies$ One-Step Consistency

Practical Stability

Practical Stability

$$\mathbf{x}_{k+1} = \mathbf{F}_h(\mathbf{x}_k, \mathbf{k}_h(\mathbf{x}_k))$$

Practical Stability

Practical Stability

$$\mathbf{x}_{k+1} = \mathbf{F}_h(\mathbf{x}_k, \mathbf{k}_h(\mathbf{x}_k))$$

$$\forall R > 0 \qquad \exists h^* > 0$$

$$h \in (0, h^*] \implies \|\mathbf{x}_k\| \leq \beta(\|\mathbf{x}_0\|, kh) + R$$

$$\beta \in \mathcal{KL}$$

Practical Stability

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$$\beta \in \mathcal{KL}$$

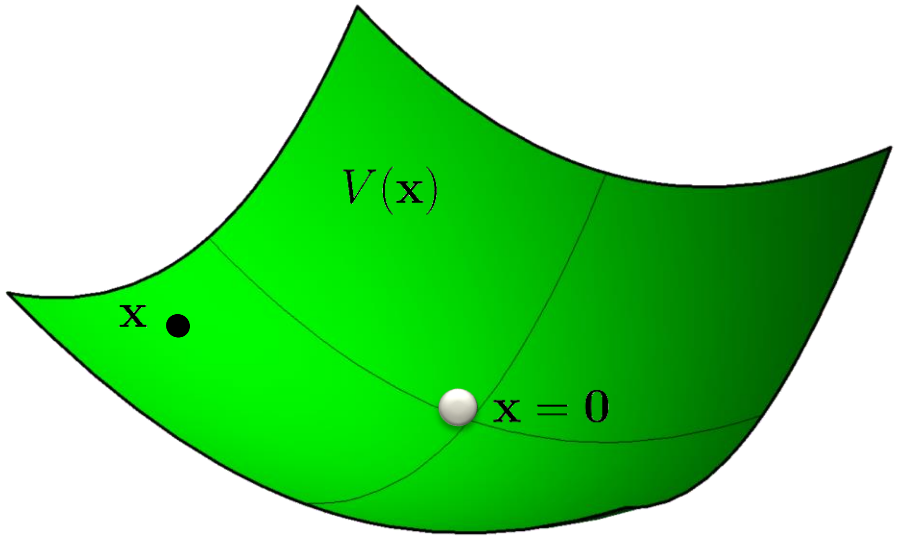
Equi-Lipschitz Lyapunov Function

$$\alpha_1(\|\mathbf{x}\|) \leq V_h(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$$

$$V_h(\mathbf{F}_h(\mathbf{x}, \mathbf{k}(\mathbf{x}))) - V_h(\mathbf{x}) \leq -h\alpha_3(\|\mathbf{x}\|)$$

$$|V_h(\mathbf{x}) - V_h(\mathbf{y})| \leq M\|\mathbf{x} - \mathbf{y}\|$$

$$\forall h \in (0, h^*] \qquad \alpha_i \in \mathcal{K}$$



Practical Stability

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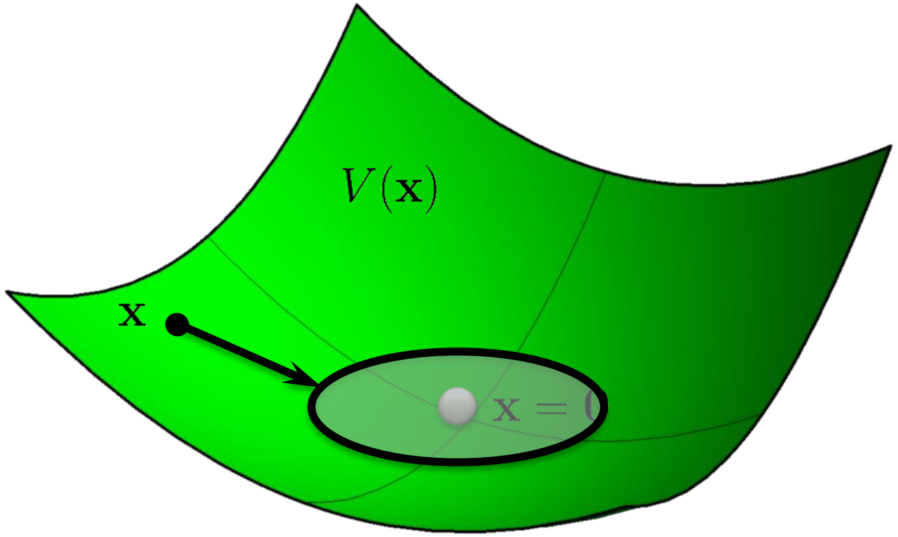

V_h for $\mathbf{F}_h \implies \mathbf{F}_h$ practically stable

Practical Stability

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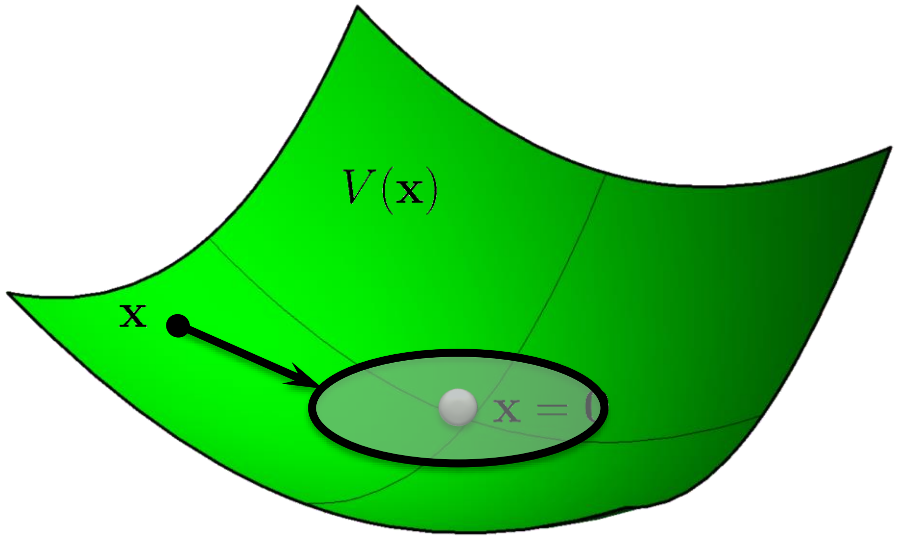

V_h for $\mathbf{F}_h \implies \mathbf{F}_h$ practically stable

V_h for $\mathbf{F}_h^a$ + One-Step Consistency $\implies \mathbf{F}_h^e$ practically stable

Practical Stability

Practical Stability

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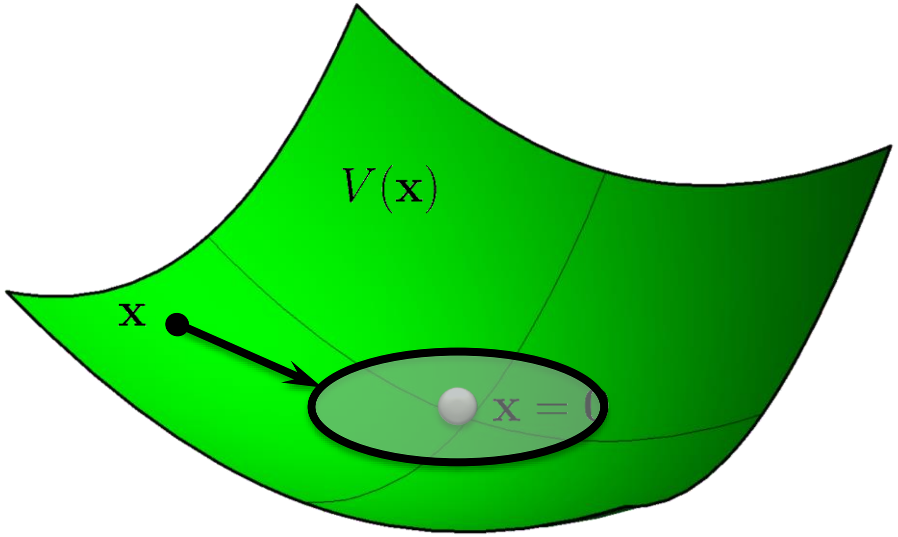
How do we design k ?



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V_h for $\mathbf{F}_h^a$ + One-Step Consistency $\implies \mathbf{F}_h^e$ practically stable

Practical Stability

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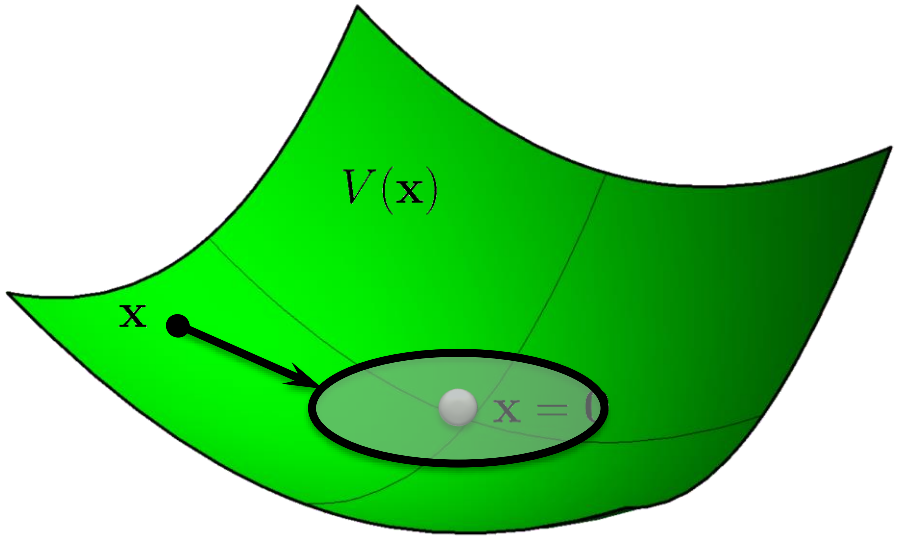
Can we use feedback linearization and CLFs?



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Feedback Linearization with Sampling

Feedback Linearization + Sampling

[9] J. W. Grizzle, "Feedback Linearization of discrete-time systems", 1986.
[10] S. Monaco, D. Normand-Cyrot, S. Stornelli, "On the linearizing feedback in nonlinear sampled data control schemes", 1986.
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Feedback and sampling do not commute

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Feedback and sampling do not commute

Feedback Linearization of Sampled-Data Systems

J. W. GRIZZLE AND P. V. KOKOTOVIC

$$\dot{x}^1 = x^2,\ \dot{x}^2 = u(1+(x^2)^2)$$

Feedback Linearization with Sampling

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Feedback Linearization of Sampled-Data Systems

J. W. GRIZZLE AND P. V. KOKOTOVIC

$$\dot{x}^1 = x^2, \ \dot{x}^2 = u(1+(x^2)^2)$$

For a sampling interval T, the sampled-data system (exact discretization) is

$$x^1_{k+1} = x^1_k + \int_0^T \tan\,(u\tau + \arctan\,(x^2_k))\, d\tau, \ x^2_{k+1} = \tan\,(uT + \arctan\,(x^2_k)). \tag{2.2}$$

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If one obtains approximating systems by performing a Taylor expansion in the sampling interval T, then, in the coordinates used above, neglecting second and higher order terms in T results in a system that is feedback linearizable, whereas neglecting only third and higher order terms results in a system that is not feedback linearizable.

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Feedback and sampling do not commute

Feedback Linearization of Sampled-Data Systems

J. W. GRIZZLE AND P. V. KOKOTOVIC

However, after the change of coordinates $y^1 = x^1 + (x^2)^2$, $y^2 = x^2$, (2.2) becomes

$$y^1_{k+1} = y^2_k - (y^2_k)^2 + \int_0^T \tan\,(u\tau + \arctan\,(y^2_k))\,d\tau + \tan^2\,(uT + \arctan\,(y^2_k))$$

$$y^2_{k+1} = \tan\,(uT + \arctan\,(y^2_k)). \tag{2.4}$$

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J. W. GRIZZLE AND P. V. KOKOTOVIC

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$$y^2_{k+1} = \tan\,(uT + \arctan\,(y^2_k)). \qquad (2.4)$$

Now, expanding the right-hand side of (2.4) and dropping terms second order and higher in T yields a system that is not feedback linearizable.

Feedback Linearization with Sampling

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Feedback Linearization + Sampling

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J. W. GRIZZLE AND P. V. KOKOTOVIC

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Now, expanding the right-hand side of (2.4) and dropping terms second order and higher in T yields a system that is not feedback linearizable.

Hence, one cannot state that the obstruction to feedback linearizability is “second order or higher in T,” without also specifying a particular set of coordinates

Feedback Linearization with Sampling

State Dynamics
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$

Feedback Linearization with Sampling

State Dynamics
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$

Exact Discrete Transition Map
$\mathbf{x}_{k+1} = \mathbf{F}_h^{e,\mathbf{x}}(\mathbf{x}_k, \mathbf{u}_k) = \mathbf{x}_k + \int_0^h \mathbf{f}(\mathbf{x}(\tau)) + \mathbf{g}(\mathbf{x}(\tau))\mathbf{u}_k \mathrm{d}\tau$

Feedback Linearization with Sampling

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State Dynamics

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Euler Approximate Discrete Transition Map

$$\mathbf{x}_{k+1} = \mathbf{F}_h^{a,\mathbf{x}}(\mathbf{x}_k, \mathbf{u}_k) = \mathbf{x}_k + h\left(\mathbf{f}(\mathbf{x}_k) + \mathbf{g}(\mathbf{x}_k)\mathbf{u}_k\right)$$

Feedback Linearization with Sampling

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Not feedback linearizable!

Feedback Linearization with Sampling

State Dynamics

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Feedback Linearization with Sampling

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State Dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \boldsymbol{\Phi}(\mathbf{x})$$

$$\boldsymbol{\Phi}(\mathbf{0}) = \mathbf{0}$$

Normal Dynamics

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\mathbf{z}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix}}_{\mathbf{f}_{\boldsymbol{\xi}}} + \underbrace{\begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{0} \end{bmatrix}}_{\mathbf{g}_{\boldsymbol{\xi}}} \mathbf{u}$$

Feedback Linearization with Sampling

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Exact Discrete Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{e,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + \int_0^h \begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}(\tau)) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}(\tau)) \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k \mathrm{d}\tau$$

Feedback Linearization with Sampling

Caltech UCLA

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Euler Approximate Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + h \left(\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}_k) \\ \mathbf{q}(\boldsymbol{\xi}_k) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}_k) \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k \right)$$

Feedback Linearization with Sampling

Caltech UCLA

State Dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \boldsymbol{\Phi}(\mathbf{x})$$

$$\boldsymbol{\Phi}(\mathbf{0}) = \mathbf{0}$$

Normal Dynamics

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\mathbf{z}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix}}_{\mathbf{f}_{\boldsymbol{\xi}}} + \underbrace{\begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{0} \end{bmatrix}}_{\mathbf{g}_{\boldsymbol{\xi}}} \mathbf{u}$$

Exact Discrete Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{e,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + \int_0^h \begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}(\tau)) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}(\tau)) \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k d\tau$$

Euler Approximate Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + h\left(\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}_k) \\ \mathbf{q}(\boldsymbol{\xi}_k) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}_k) \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k \right)$$

$$\mathbf{u}_k = \mathbf{k}_{\mathrm{fbl}}(\boldsymbol{\xi}_k)$$

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + h \begin{bmatrix} \mathbf{A}\boldsymbol{\eta}_k \\ \mathbf{q}(\boldsymbol{\xi}_k) \end{bmatrix}$$

Feedback Linearization with Sampling

State Dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \boldsymbol{\Phi}(\mathbf{x})$$

$$\boldsymbol{\Phi}(\mathbf{0}) = \mathbf{0}$$

Normal Dynamics

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\mathbf{z}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix}}_{\mathbf{f}_{\boldsymbol{\xi}}} + \underbrace{\begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{0} \end{bmatrix}}_{\mathbf{g}_{\boldsymbol{\xi}}} \mathbf{u}$$

Exact Discrete Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{e,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + \int_0^h \begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}(\tau)) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}(\tau)) \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k d\tau$$

Euler Approximate Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + h\left(\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}_k) \\ \mathbf{q}(\boldsymbol{\xi}_k) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}_k) \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k \right)$$

$$\mathbf{u}_k = \mathbf{k}_{\mathrm{fbl}}(\boldsymbol{\xi}_k)$$

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \begin{bmatrix} (\mathbf{I} + h\mathbf{A})\boldsymbol{\eta}_k \\ \mathbf{z}_k + h\mathbf{q}(\boldsymbol{\xi}_k) \end{bmatrix}$$

Feedback Linearization with Sampling

Transform then sample!

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State Dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \boldsymbol{\Phi}(\mathbf{x})$$

$$\boldsymbol{\Phi}(\mathbf{0}) = \mathbf{0}$$

Normal Dynamics

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\mathbf{z}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix}}_{\mathbf{f}_{\boldsymbol{\xi}}} + \underbrace{\begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) \\ \mathbf{0} \end{bmatrix}}_{\mathbf{g}_{\boldsymbol{\xi}}} \mathbf{u}$$

Exact Discrete Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{e,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + \int_0^h \begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}(\tau)) \\ \mathbf{q}(\boldsymbol{\xi}) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}(\tau)) \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k d\tau$$

Euler Approximate Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \boldsymbol{\eta}_k \\ \mathbf{z}_k \end{bmatrix} + h\left(\begin{bmatrix} \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}_k) \\ \mathbf{q}(\boldsymbol{\xi}_k) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi}_k) \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k \right)$$

$$\mathbf{u}_k = \mathbf{k}_{\mathrm{fbl}}(\boldsymbol{\xi}_k)$$

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \begin{bmatrix} (\mathbf{I} + h\mathbf{A})\boldsymbol{\eta}_k \\ \mathbf{z}_k + h\mathbf{q}(\boldsymbol{\xi}_k) \end{bmatrix}$$

Discrete CLF Synthesis

Lyapunov Equation

$$\mathbf{A}^\top \mathbf{P}_\eta + \mathbf{P}_\eta \mathbf{A} = -\mathbf{Q}_\eta$$
$$\mathbf{P}_\eta, \mathbf{Q}_\eta \succ 0$$

Discrete CLF Synthesis

Lyapunov Equation

$$\mathbf{A}^\top \mathbf{P}_\eta + \mathbf{P}_\eta \mathbf{A} = -\mathbf{Q}_\eta$$
$$\mathbf{P}_\eta, \mathbf{Q}_\eta \succ \mathbf{0}$$

$\Downarrow$

Lyapunov Function

$$V_\eta(\boldsymbol{\eta}) = \boldsymbol{\eta}^\top \mathbf{P}_\eta \boldsymbol{\eta}$$

Discrete CLF Synthesis

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Lyapunov Equation

$$\mathbf{A}^\top \mathbf{P}_{\boldsymbol{\eta}} + \mathbf{P}_{\boldsymbol{\eta}} \mathbf{A} = -\mathbf{Q}_{\boldsymbol{\eta}}$$
$$\mathbf{P}_{\boldsymbol{\eta}}, \mathbf{Q}_{\boldsymbol{\eta}} \succ \mathbf{0}$$

Lyapunov Function

$$V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) = \boldsymbol{\eta}^\top \mathbf{P}_{\boldsymbol{\eta}} \boldsymbol{\eta}$$

Approximate Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}_k, \mathbf{u}_k) \\ \mathbf{F}_h^{a,\mathbf{z}}(\boldsymbol{\xi}_k, \mathbf{u}_k) \end{bmatrix}$$

Discrete CLF Synthesis

Lyapunov Equation

$$\mathbf{A}^\top \mathbf{P}_{\boldsymbol{\eta}} + \mathbf{P}_{\boldsymbol{\eta}} \mathbf{A} = -\mathbf{Q}_{\boldsymbol{\eta}}$$
$$\mathbf{P}_{\boldsymbol{\eta}}, \mathbf{Q}_{\boldsymbol{\eta}} \succ \mathbf{0}$$

Approximate Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}_k, \mathbf{u}_k) \\ \mathbf{F}_h^{a,\mathbf{z}}(\boldsymbol{\xi}_k, \mathbf{u}_k) \end{bmatrix}$$

Lyapunov Function

$$V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) = \boldsymbol{\eta}^\top \mathbf{P}_{\boldsymbol{\eta}} \boldsymbol{\eta}$$

Discrete Control Lyapunov Function

$\forall c \in (0,1) \ \ \exists h^* > 0$

$$\lambda_{\min}(\mathbf{P}_{\boldsymbol{\eta}}) \|\boldsymbol{\eta}\|_2^2 \leq V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) \leq \lambda_{\max}(\mathbf{P}_{\boldsymbol{\eta}}) \|\boldsymbol{\eta}\|_2^2$$

$$\inf_{\mathbf{u} \in \mathbb{R}^m} V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}, \mathbf{u})) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}}) \|\boldsymbol{\eta}\|_2^2$$
$$\forall h \in (0, h^*]$$

Discrete CLF Synthesis

Lyapunov Equation

$$\mathbf{A}^\top \mathbf{P}_{\boldsymbol{\eta}} + \mathbf{P}_{\boldsymbol{\eta}} \mathbf{A} = -\mathbf{Q}_{\boldsymbol{\eta}}$$
$$\mathbf{P}_{\boldsymbol{\eta}}, \mathbf{Q}_{\boldsymbol{\eta}} \succ \mathbf{0}$$

Lyapunov Function

$$V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) = \boldsymbol{\eta}^\top \mathbf{P}_{\boldsymbol{\eta}} \boldsymbol{\eta}$$

Uniformity properties!

Approximate Transition Map

$$\begin{bmatrix} \boldsymbol{\eta}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{u}_k) = \begin{bmatrix} \mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}_k, \mathbf{u}_k) \\ \mathbf{F}_h^{a,\mathbf{z}}(\boldsymbol{\xi}_k, \mathbf{u}_k) \end{bmatrix}$$

Discrete Control Lyapunov Function

$$\forall c \in (0,1) \quad \exists h^* > 0$$

$$\lambda_{\min}(\mathbf{P}_{\boldsymbol{\eta}}) \|\boldsymbol{\eta}\|_2^2 \leq V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) \leq \lambda_{\max}(\mathbf{P}_{\boldsymbol{\eta}}) \|\boldsymbol{\eta}\|_2^2$$

$$\inf_{\mathbf{u} \in \mathbb{R}^m} V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}, \mathbf{u})) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}}) \|\boldsymbol{\eta}\|_2^2$$

$$\forall h \in (0, h^*]$$

Sampled-Data Stabilization

Output Stabilization

$$V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}_k, \mathbf{k}_h(\boldsymbol{\xi}_k))) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}_k) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}})\|\boldsymbol{\eta}_k\|_2^2$$

Sampled-Data Stabilization

Zero Dynamics

$$\dot{\mathbf{z}} = \mathbf{q}(\mathbf{0}, \mathbf{z})$$

Output Stabilization

$$V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}_k, \mathbf{k}_h(\boldsymbol{\xi}_k))) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}_k) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}})\|\boldsymbol{\eta}_k\|_2^2$$

Sampled-Data Stabilization

Zero Dynamics

$$\dot{\mathbf{z}} = \mathbf{q}(\mathbf{0}, \mathbf{z})$$

Local Exponential Stability

Lyapunov Function

$$V_{\mathbf{z}}(\mathbf{z}) = \mathbf{z}^\top \mathbf{P}_{\mathbf{z}} \mathbf{z}$$

Output Stabilization

$$V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}_k, \mathbf{k}_h(\boldsymbol{\xi}_k))) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}_k) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}})\|\boldsymbol{\eta}_k\|_2^2$$

Sampled-Data Stabilization

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Zero Dynamics

$$\dot{\mathbf{z}} = \mathbf{q}(\mathbf{0}, \mathbf{z})$$

Local Exponential Stability

Lyapunov Function

$$V_{\mathbf{z}}(\mathbf{z}) = \mathbf{z}^\top \mathbf{P}_{\mathbf{z}}\mathbf{z}$$

Uniformity Properties

$$\forall d \in (0,1) \;\; \exists h^* > 0$$

$$\lambda_{\min}(\mathbf{P}_{\mathbf{z}})\|\mathbf{z}\|_2^2 \leq V_{\mathbf{z}}(\mathbf{z}) \leq \lambda_{\max}(\mathbf{P}_{\mathbf{z}})\|\mathbf{z}\|_2^2$$

$$V_{\mathbf{z}}(\mathbf{F}_h^{a,\mathbf{z}}(\boldsymbol{\xi})) - V_{\mathbf{z}}(\mathbf{z}) \leq -hd\lambda_{\min}(\mathbf{Q}_{\mathbf{z}})\|\mathbf{z}\|_2^2$$

$$\forall h \in (0, h^*]$$

Output Stabilization

$$V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}_k, \mathbf{k}_h(\boldsymbol{\xi}_k))) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}_k) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}})\|\boldsymbol{\eta}_k\|_2^2$$

Sampled-Data Stabilization

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Zero Dynamics

$$\dot{\mathbf{z}} = \mathbf{q}(\mathbf{0}, \mathbf{z})$$

Local Exponential Stability

Lyapunov Function

$$V_{\mathbf{z}}(\mathbf{z}) = \mathbf{z}^\top \mathbf{P}_{\mathbf{z}} \mathbf{z}$$

Uniformity Properties

$$\forall d \in (0,1) \;\; \exists h^* > 0$$
$$\lambda_{\min}(\mathbf{P}_{\mathbf{z}}) \|\mathbf{z}\|_2^2 \leq V_{\mathbf{z}}(\mathbf{z}) \leq \lambda_{\max}(\mathbf{P}_{\mathbf{z}}) \|\mathbf{z}\|_2^2$$
$$V_{\mathbf{z}}(\mathbf{F}_h^{a,\mathbf{z}}(\boldsymbol{\xi})) - V_{\mathbf{z}}(\mathbf{z}) \leq -hd\lambda_{\min}(\mathbf{Q}_{\mathbf{z}}) \|\mathbf{z}\|_2^2$$
$$\forall h \in (0, h^*]$$

Output Stabilization

$$V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}_k, \mathbf{k}_h(\boldsymbol{\xi}_k))) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}_k) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}}) \|\boldsymbol{\eta}_k\|_2^2$$

Composite Lyapunov

$$V(\boldsymbol{\xi}) = \sigma V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) + V_{\mathbf{z}}(\mathbf{z})$$
$$\sigma > 0$$

Sampled-Data Stabilization

Composite Lyapunov

$$V(\boldsymbol{\xi}) = \sigma V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) + V_{\mathbf{z}}(\mathbf{z})$$

$$\sigma > 0$$

Sampled-Data Stabilization

Composite Lyapunov

$$V(\boldsymbol{\xi}) = \sigma V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) + V_{\mathbf{z}}(\mathbf{z})$$
$$\sigma > 0$$

Choice of σ

Uniformity Properties

$$\exists h^* > 0$$
$$k_1 \|\boldsymbol{\xi}\|_2^2 \leq V(\boldsymbol{\xi}) \leq k_2 \|\boldsymbol{\xi}\|_2^2$$
$$V(\mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi})) - V(\boldsymbol{\xi}) \leq -hk_3 \|\boldsymbol{\xi}\|_2^2$$
$$|V(\boldsymbol{\xi}_2) - V(\boldsymbol{\xi}_1)| \leq M \|\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1\|$$
$$\forall h \in (0, h^*] \qquad \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in K$$

Sampled-Data Stabilization

Composite Lyapunov

$$V(\boldsymbol{\xi}) = \sigma V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) + V_{\mathbf{z}}(\mathbf{z})$$
$$\sigma > 0$$

Choice of σ

Uniformity Properties

$$\exists h^* > 0$$
$$k_1\|\boldsymbol{\xi}\|_2^2 \leq V(\boldsymbol{\xi}) \leq k_2\|\boldsymbol{\xi}\|_2^2$$
$$V(\mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi})) - V(\boldsymbol{\xi}) \leq -hk_3\|\boldsymbol{\xi}\|_2^2$$
$$|V(\boldsymbol{\xi}_2) - V(\boldsymbol{\xi}_1)| \leq M\|\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1\|$$
$$\forall h \in (0, h^*] \qquad \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in K$$

Practical Stability

$$\boldsymbol{\xi}_{k+1} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{k}(\boldsymbol{\xi}_k))$$

Sampled-Data Stabilization

Composite Lyapunov

$$V(\boldsymbol{\xi}) = \sigma V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) + V_{\mathbf{z}}(\mathbf{z})$$
$$\sigma > 0$$

Choice of σ

Uniformity Properties

$$\exists h^* > 0$$
$$k_1\|\boldsymbol{\xi}\|_2^2 \leq V(\boldsymbol{\xi}) \leq k_2\|\boldsymbol{\xi}\|_2^2$$
$$V(\mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi})) - V(\boldsymbol{\xi}) \leq -hk_3\|\boldsymbol{\xi}\|_2^2$$
$$|V(\boldsymbol{\xi}_2) - V(\boldsymbol{\xi}_1)| \leq M\|\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1\|$$
$$\forall h \in (0, h^*] \qquad \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in K$$

Practical Stability

$$\boldsymbol{\xi}_{k+1} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{k}(\boldsymbol{\xi}_k))$$

One-Step Consistency

Practical Stability

$$\boldsymbol{\xi}_{k+1} = \mathbf{F}_h^{e,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{k}(\boldsymbol{\xi}_k))$$

Sampled-Data Stabilization

Composite Lyapunov

$$V(\boldsymbol{\xi}) = \sigma V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) + V_{\mathbf{z}}(\mathbf{z})$$
$$\sigma > 0$$

Choice of σ

Uniformity Properties

$$\exists h^* > 0$$
$$k_1\|\boldsymbol{\xi}\|_2^2 \leq V(\boldsymbol{\xi}) \leq k_2\|\boldsymbol{\xi}\|_2^2$$
$$V(\mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi})) - V(\boldsymbol{\xi}) \leq -hk_3\|\boldsymbol{\xi}\|_2^2$$
$$|V(\boldsymbol{\xi}_2) - V(\boldsymbol{\xi}_1)| \leq M\|\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1\|$$
$$\forall h \in (0, h^*] \qquad \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in K$$

Practical Stability

$$\boldsymbol{\xi}_{k+1} = \mathbf{F}_h^{a,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{k}(\boldsymbol{\xi}_k))$$

One-Step Consistency

Practical Stability

$$\boldsymbol{\xi}_{k+1} = \mathbf{F}_h^{e,\boldsymbol{\xi}}(\boldsymbol{\xi}_k, \mathbf{k}(\boldsymbol{\xi}_k))$$

Diffeomorphism

Practical Stability

$$\mathbf{x}_{k+1} = \mathbf{F}_h^{e,\mathbf{x}}(\mathbf{x}_k, \mathbf{k}(\boldsymbol{\Phi}(\mathbf{x}_k)))$$

Quadratically Constrained Quadratic Programs

Can we do better than feedback linearization?

Quadratically Constrained Quadratic Programs

Can we do better than feedback linearization?

Discrete CLF Property

$$\inf_{\mathbf{u} \in \mathbb{R}^m} V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi}, \mathbf{u})) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}})\|\boldsymbol{\eta}\|_2^2$$

Quadratically Constrained Quadratic Programs

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Can we do better than feedback linearization?

Optimization Constraint

$$\mathbf{k}_h^{\mathrm{qcqp}}(\boldsymbol{\xi}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2$$

$$\text{s.t.}\quad V_{\boldsymbol{\eta}}(\mathbf{F}_h^{a,\boldsymbol{\eta}}(\boldsymbol{\xi},\mathbf{u})) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}) \leq -hc\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}})\|\boldsymbol{\eta}\|_2^2$$

Quadratically Constrained Quadratic Programs

Can we do better than feedback linearization?

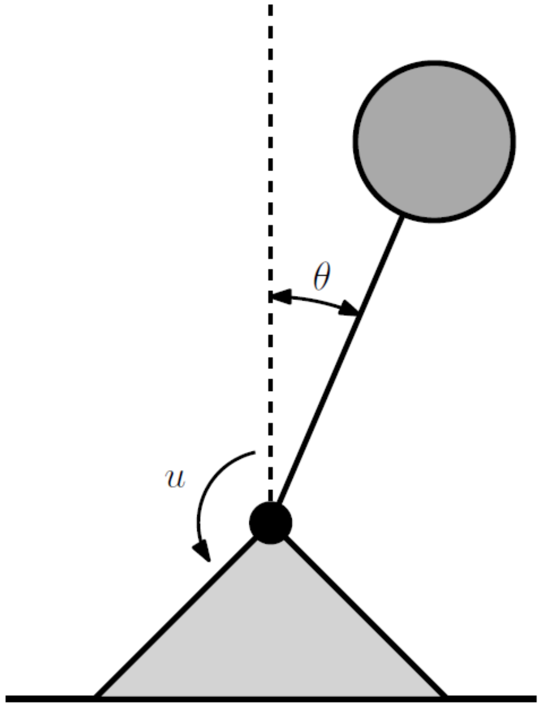
CLF Quadratically Constrained Quadratic Program

(CLF-QCQP)

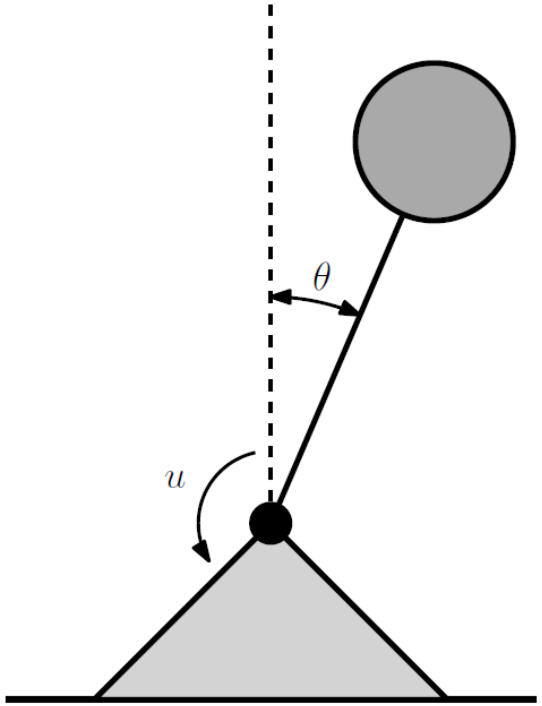
$$\mathbf{k}_h^{\mathrm{qcqp}}(\boldsymbol{\xi}) = \underset{\mathbf{u}\,\in\,\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2$$
$$\text{s.t.}\quad \mathbf{u}^\top \boldsymbol{\Lambda}_h(\boldsymbol{\xi})\mathbf{u} + 2\boldsymbol{\lambda}_h(\boldsymbol{\xi})^\top \mathbf{u} + \ell_h(\boldsymbol{\xi}) \leq 0$$

$$\boldsymbol{\Lambda}_h(\boldsymbol{\xi}) = h\mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi})^\top \mathbf{P}_{\boldsymbol{\eta}} \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi})$$
$$\boldsymbol{\lambda}_h(\boldsymbol{\xi}) = \mathbf{g}_{\boldsymbol{\eta}}(\boldsymbol{\xi})^\top \mathbf{P}_{\boldsymbol{\eta}}(\boldsymbol{\eta} + h\mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi}))$$
$$\ell_h(\boldsymbol{\xi}) = \mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi})^\top \mathbf{P}_{\boldsymbol{\eta}}(2\boldsymbol{\eta} + h\mathbf{f}_{\boldsymbol{\eta}}(\boldsymbol{\xi})) + c\lambda_{\min}(\mathbf{Q}_{\boldsymbol{\eta}})\|\boldsymbol{\eta}\|_2^2$$

Inverted Pendulum



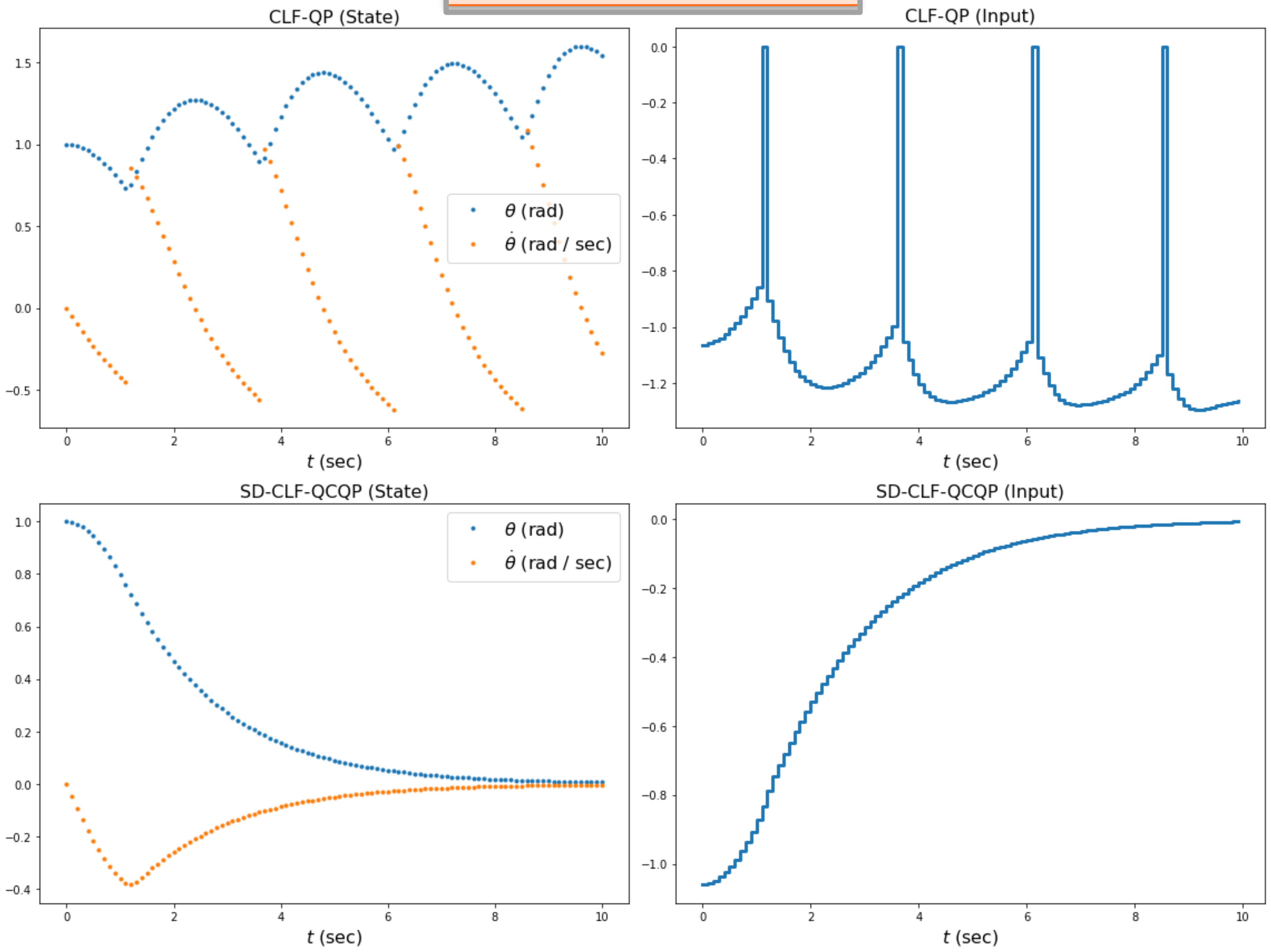
Inverted Pendulum



CLF-QP

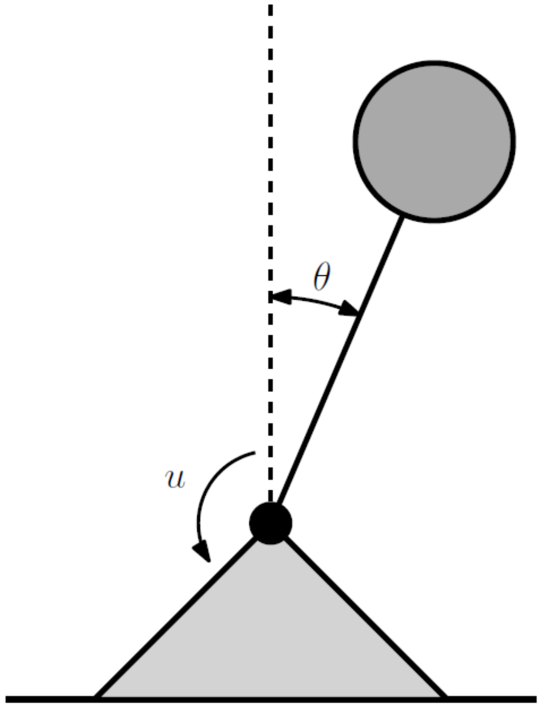
CLF-QCQP

10 Hz Simulation



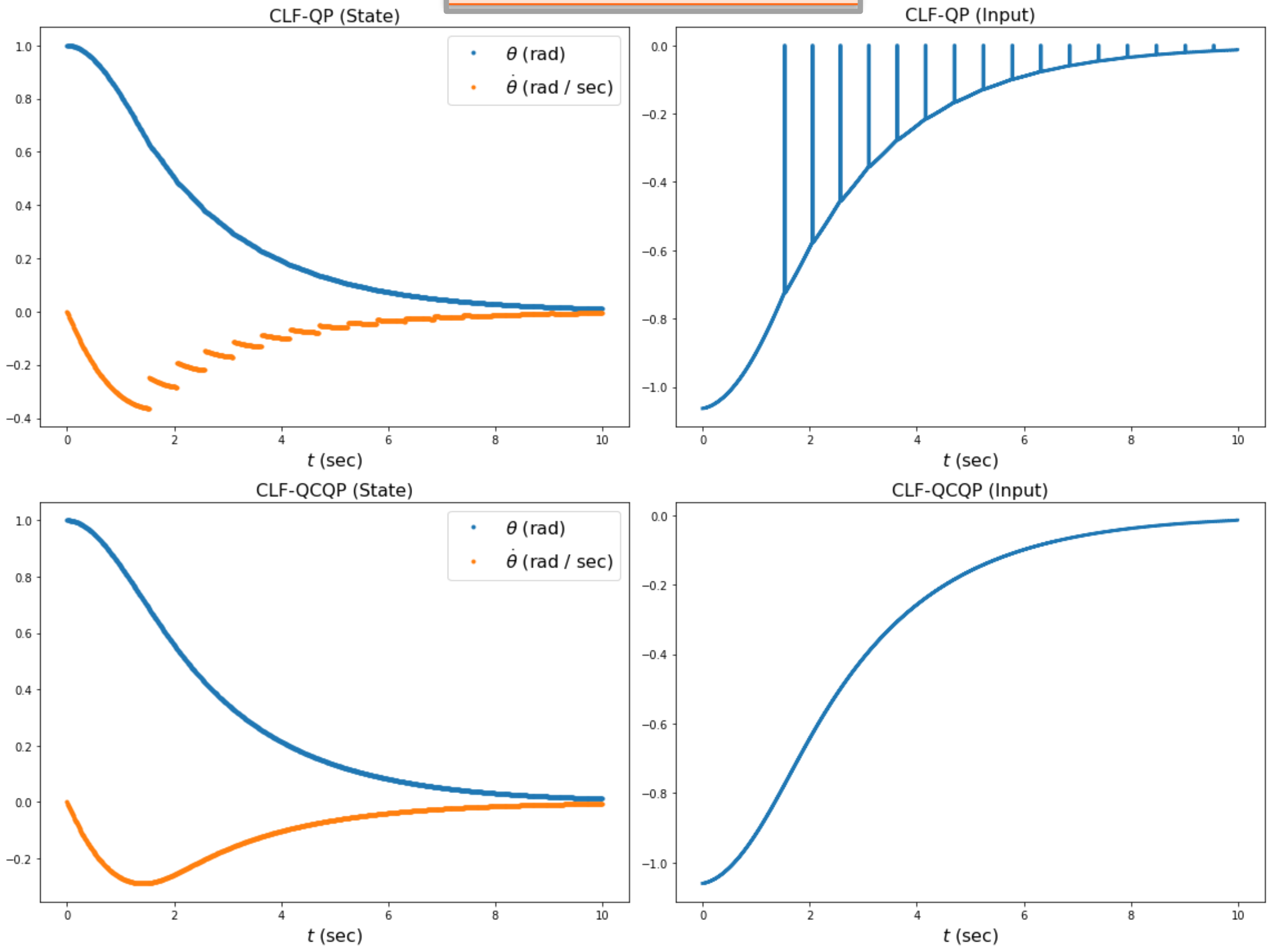
Inverted Pendulum

100 Hz Simulation

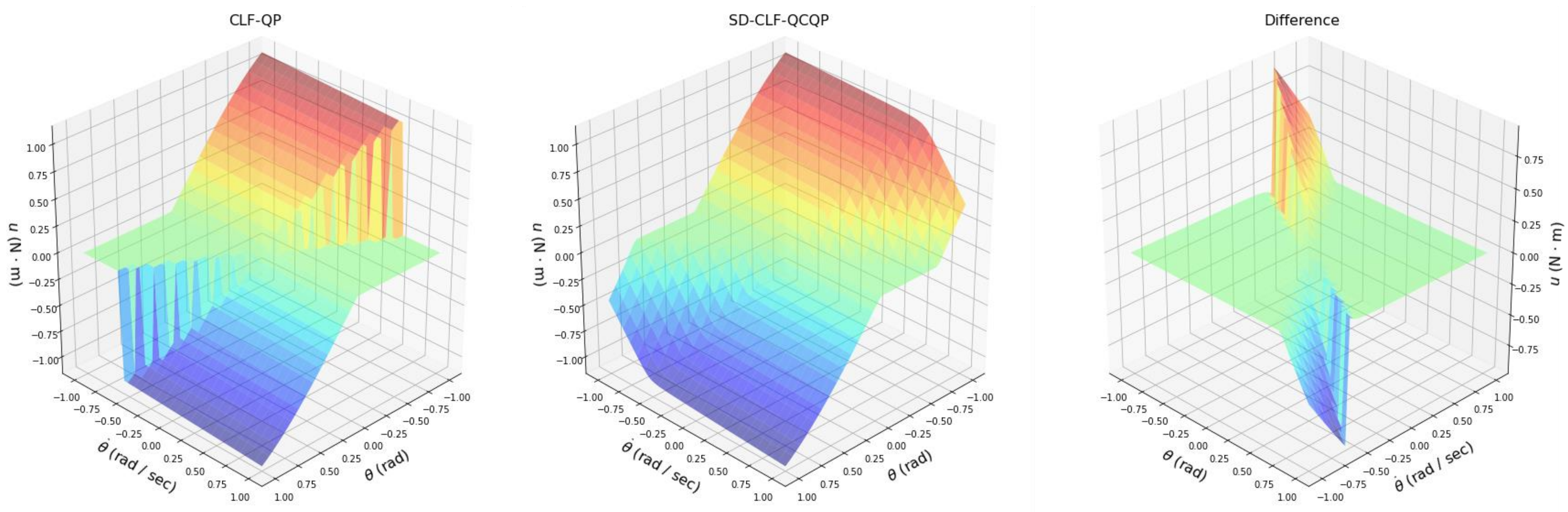


CLF-QP

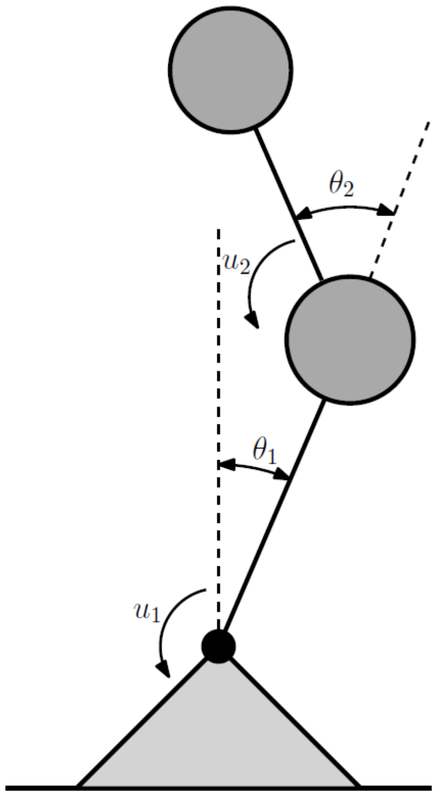
CLF-QCQP



Inverted Pendulum



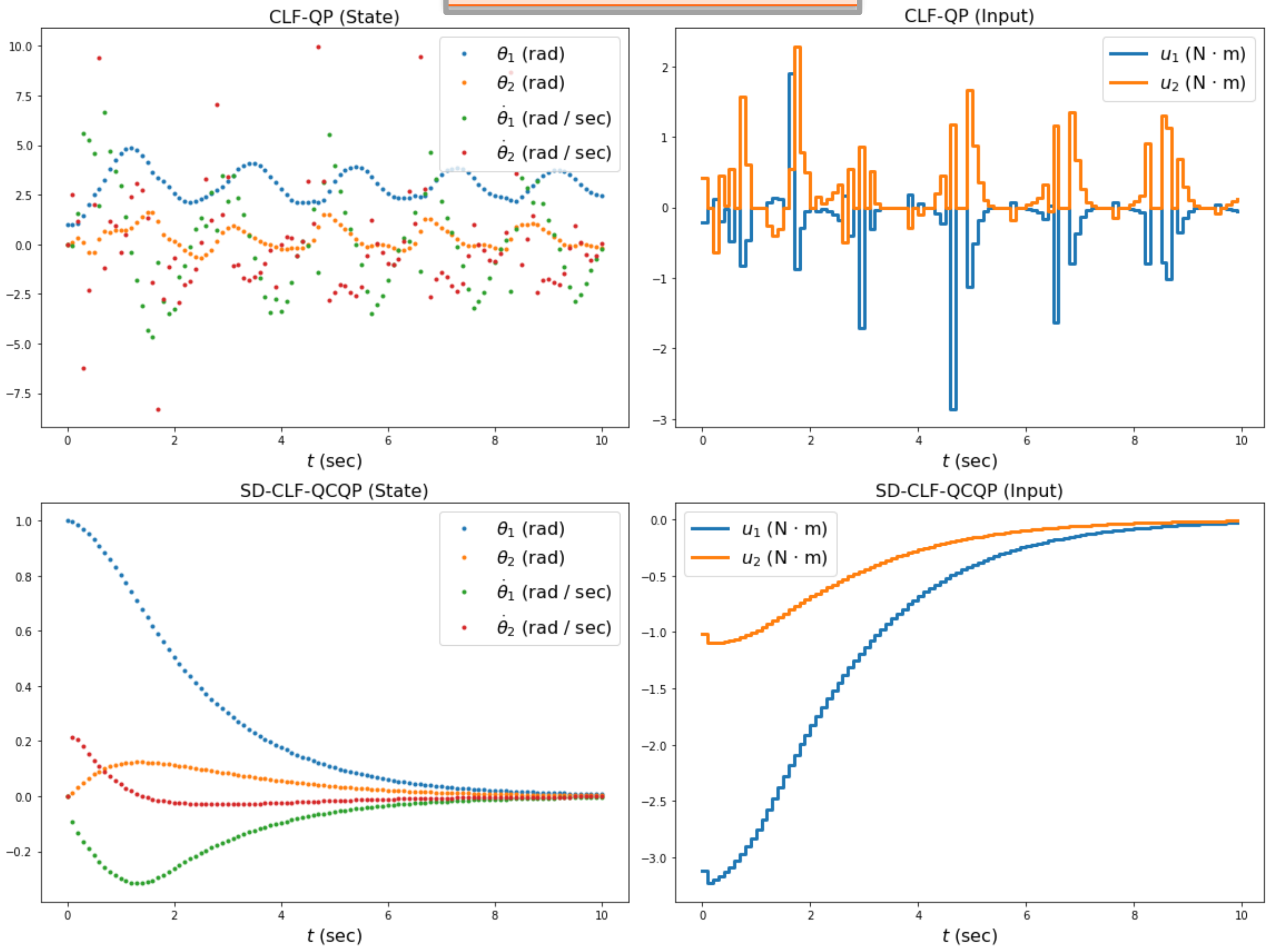
Double Inverted Pendulum



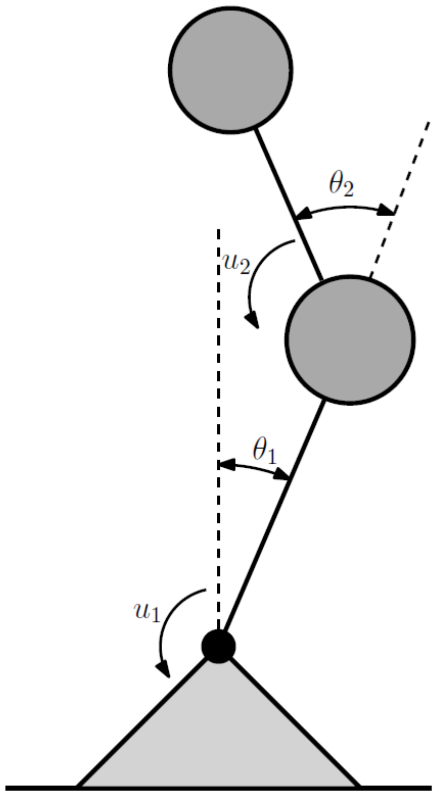
CLF-QP

CLF-QCQP

10 Hz Simulation



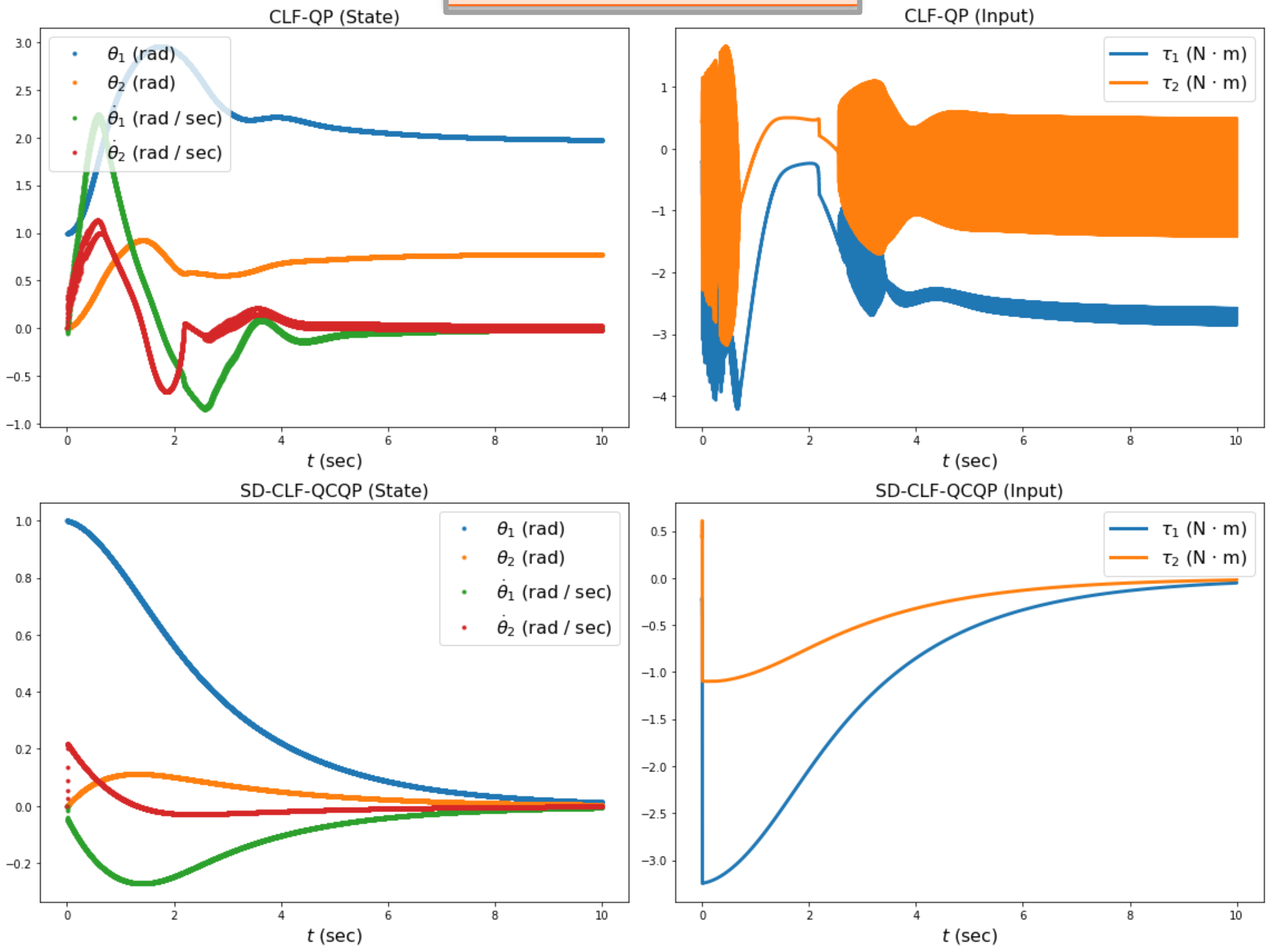
Double Inverted Pendulum



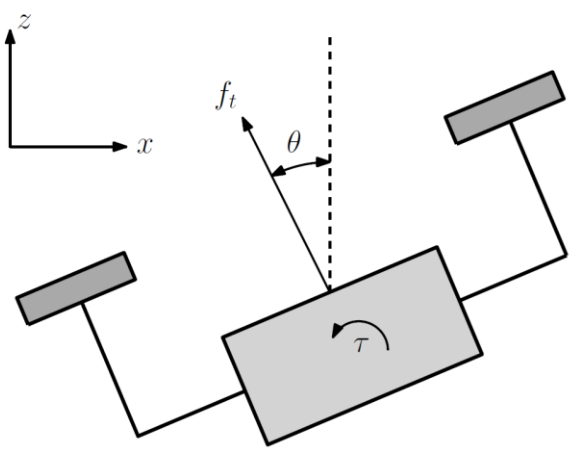
CLF-QP

CLF-QCQP

1 kHz Simulation



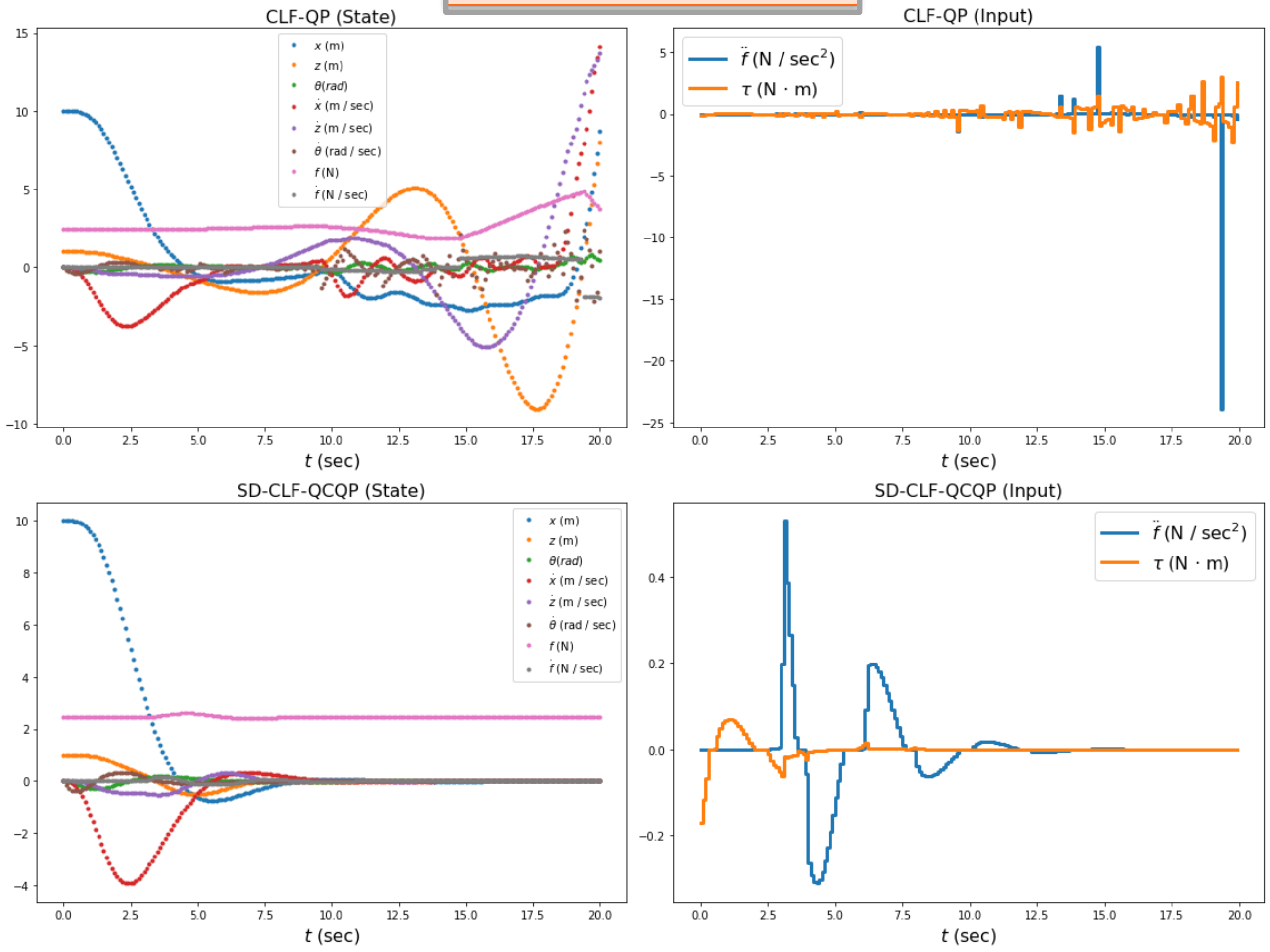
Dynamically Extended Planar Quadrotor



CLF-QP

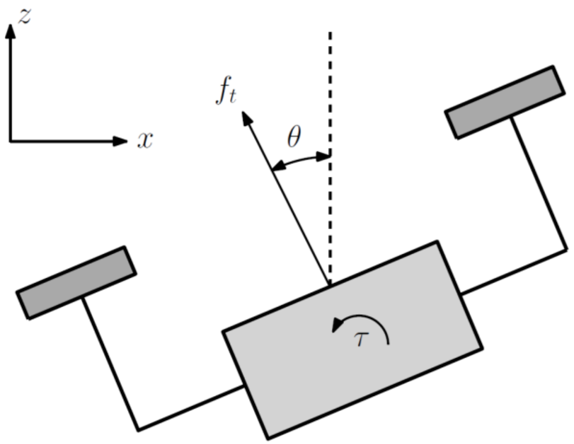
CLF-QCQP

10 Hz Simulation



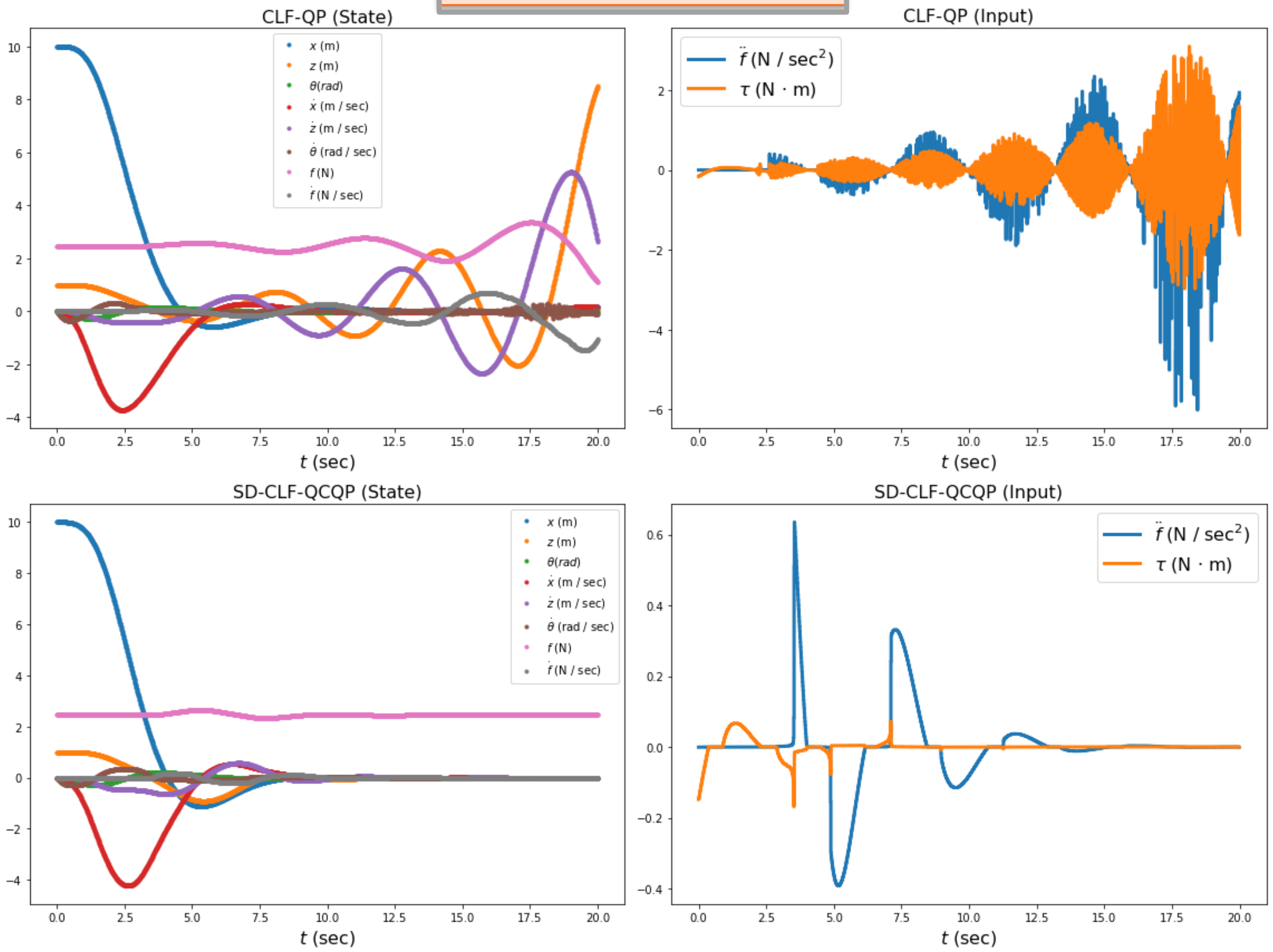
Dynamically Extended Planar Quadrotor

100 Hz Simulation



CLF-QP

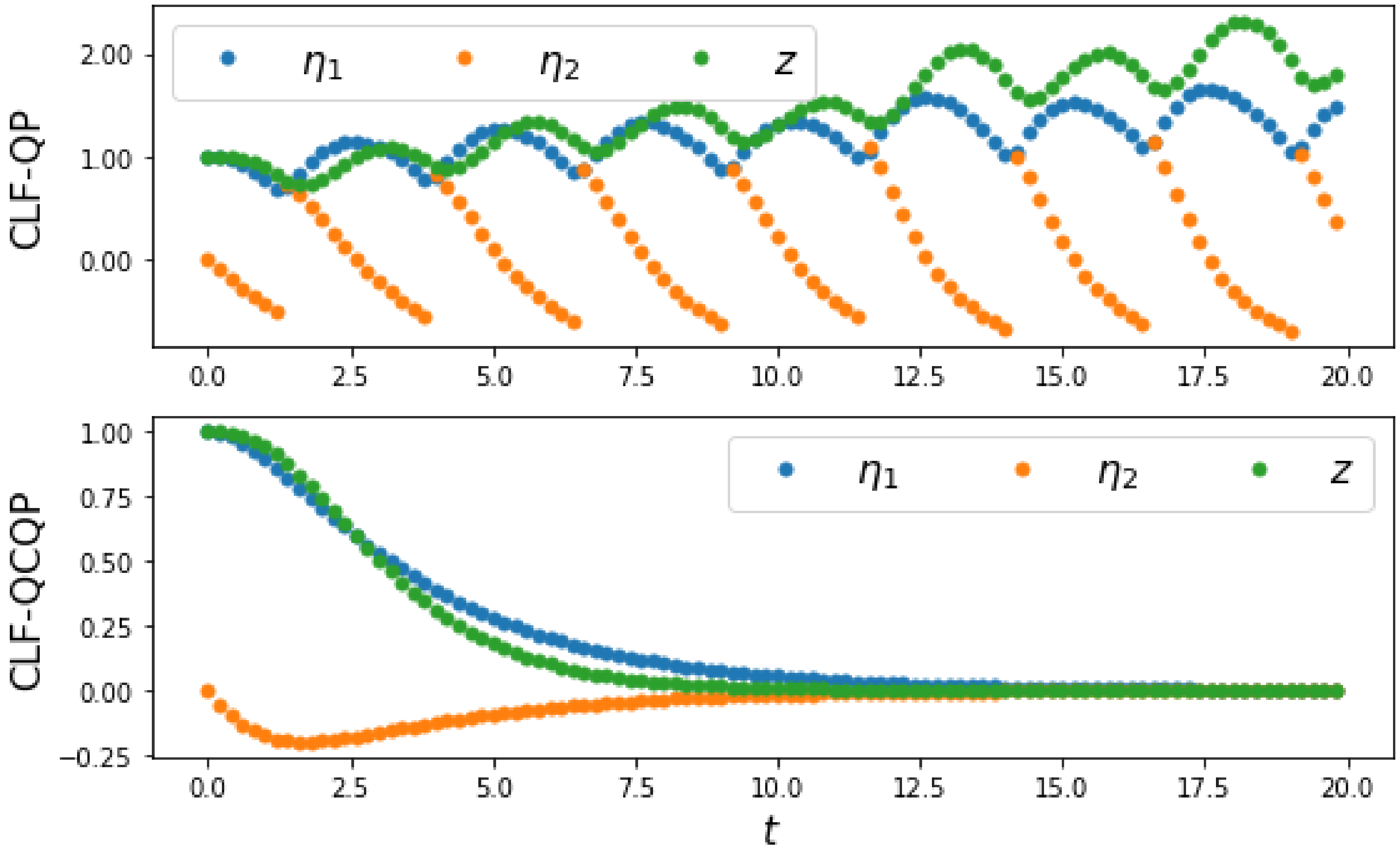
CLF-QCQP



Inverted Pendulum + Zero Dynamics

Dynamics

$$\dot{\eta}_1 = \eta_2$$
$$\dot{\eta}_2 = 10\sin(\eta_1) + u$$
$$\dot{z} = \eta_1^2 - z$$



Conclusions

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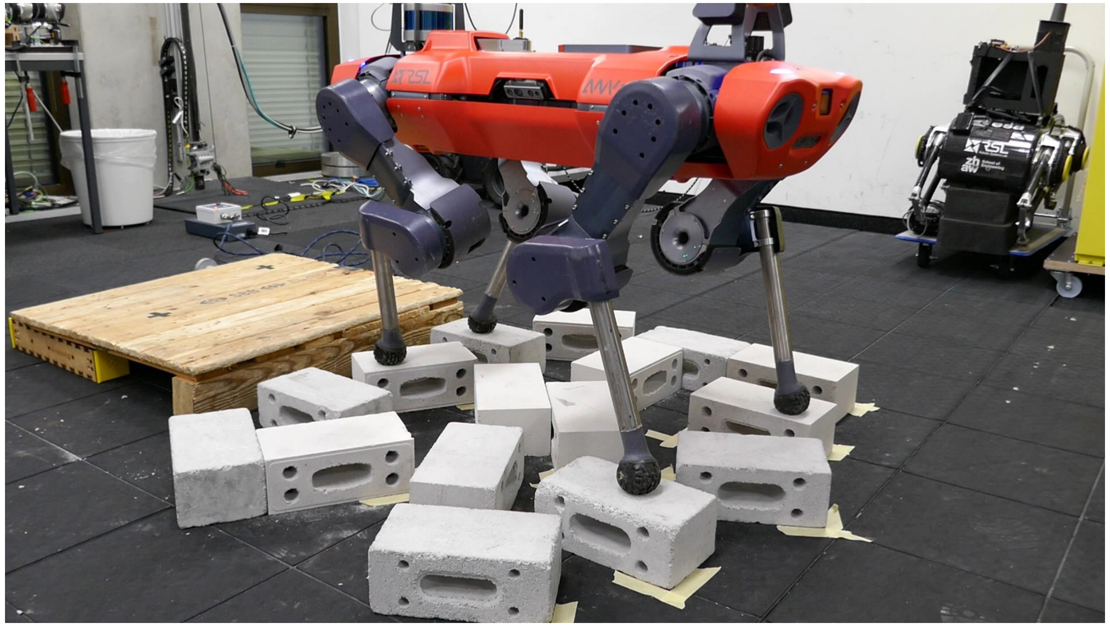
- **CLF-QCQPs** offer solution for resource efficient stabilization of sampled-data systems.
- Stability properties of zero-dynamics are preserved with sample-hold control implementation.
- CLF-QCQP displays significant improvements over CLF-QP even below theoretical frequency requirements.



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Thank You!

Sampled-Data Stabilization with Control Lyapunov Functions via Quadratically Constrained Quadratic Programs

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