Safety-Critical Event Triggered Control via Input-to-State Safe Barrier Functions

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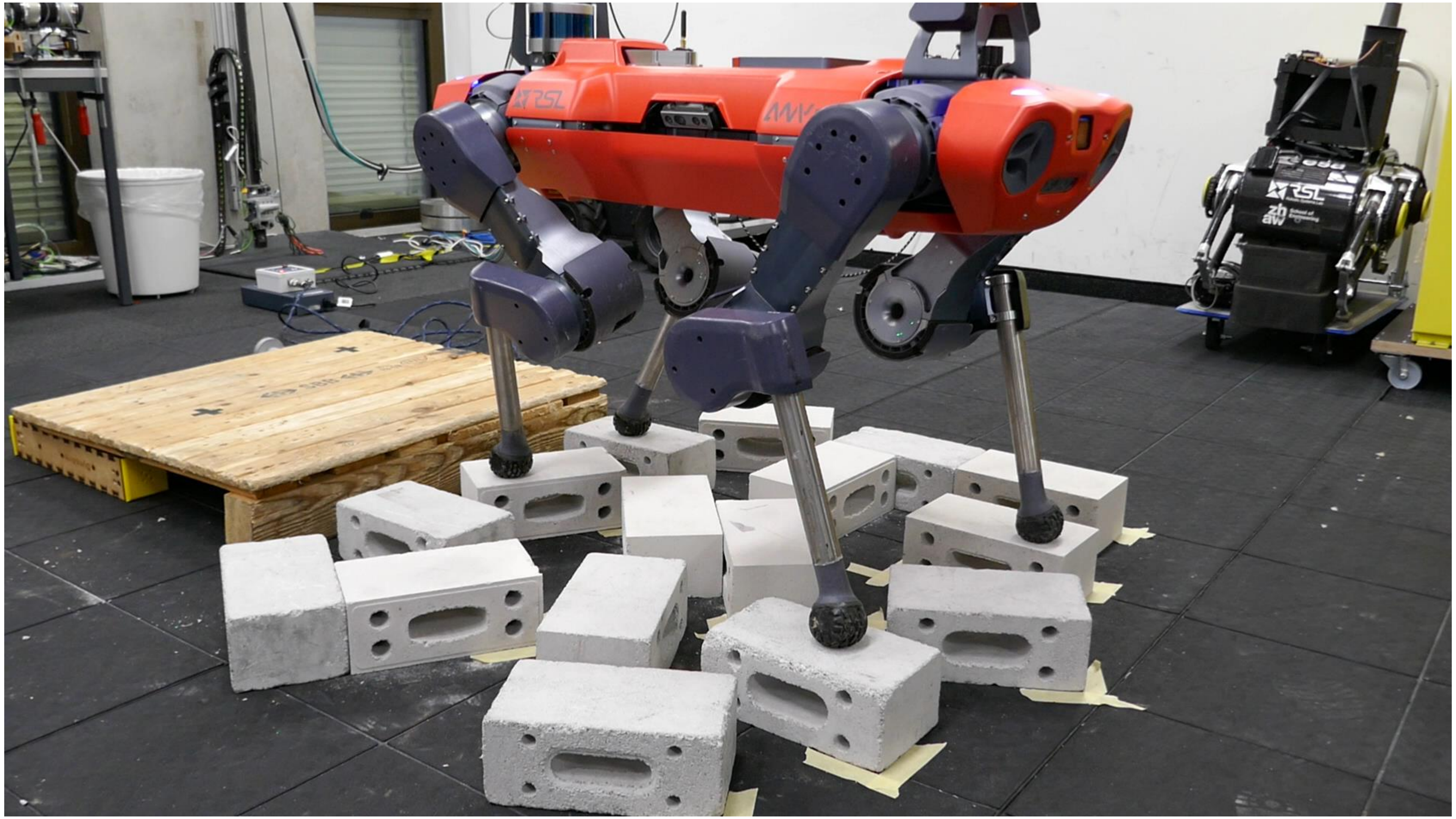
University of California at San Diego

December 17th, 2020

Control & Decision Conference (CDC) 2020

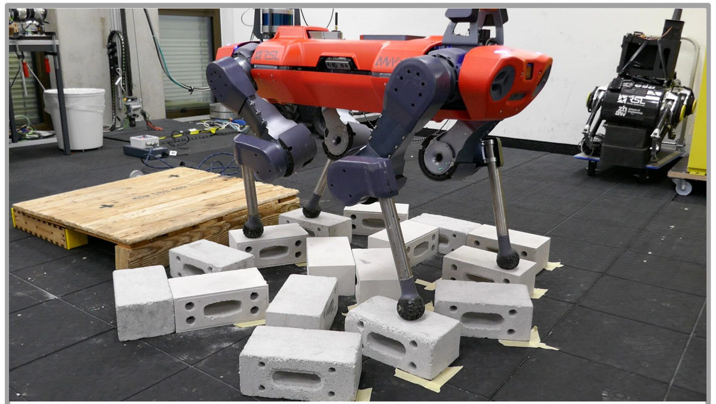
Control in the real world is hard

But: Pretty when it works…



[1] R. Grandia, **A. J. Taylor**, M. Hutter, A. D. Ames, Multi-Layered Safety for Legged Robotics via Control Barrier Functions and Model Predictive Control, 2020.

Claim: Need to Bridge the Gap



$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2$$
$$\text{s.t.} \quad \dot{h}(\mathbf{x}, \mathbf{u}) \geq -\alpha(h(\mathbf{x}))$$

Theorems & Proofs

Bridge the Gap

$$\dot{h}(\mathbf{x}, \mathbf{k}(\mathbf{x} + \mathbf{e})) \geq -\alpha(h(\mathbf{x})) - \iota(\|\mathbf{e}\|_2)$$

Experimental Realization

Contributions

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- Framework for achieving event triggered control for system safety via **Input-to-State Safe Barrier Functions (ISSf-BFs)**
- Analysis of changes in event triggered conditions from stability to safety through a pathological example
- Evaluation of minimum interevent time (MIET) using ISSf-BF trigger law

System Dynamics

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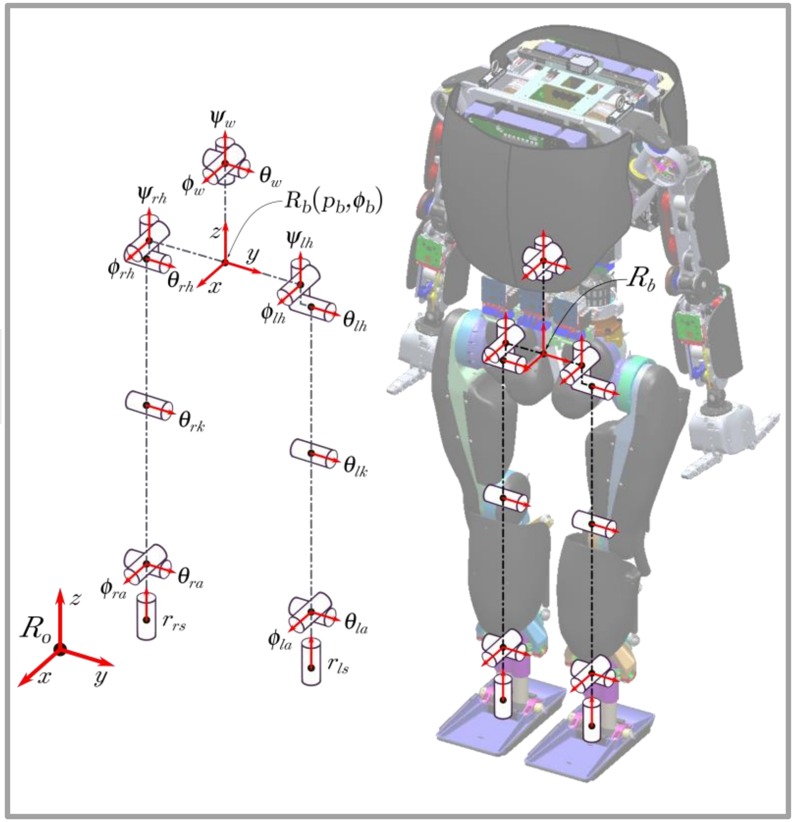
Equations of Motion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

Mathematical Model



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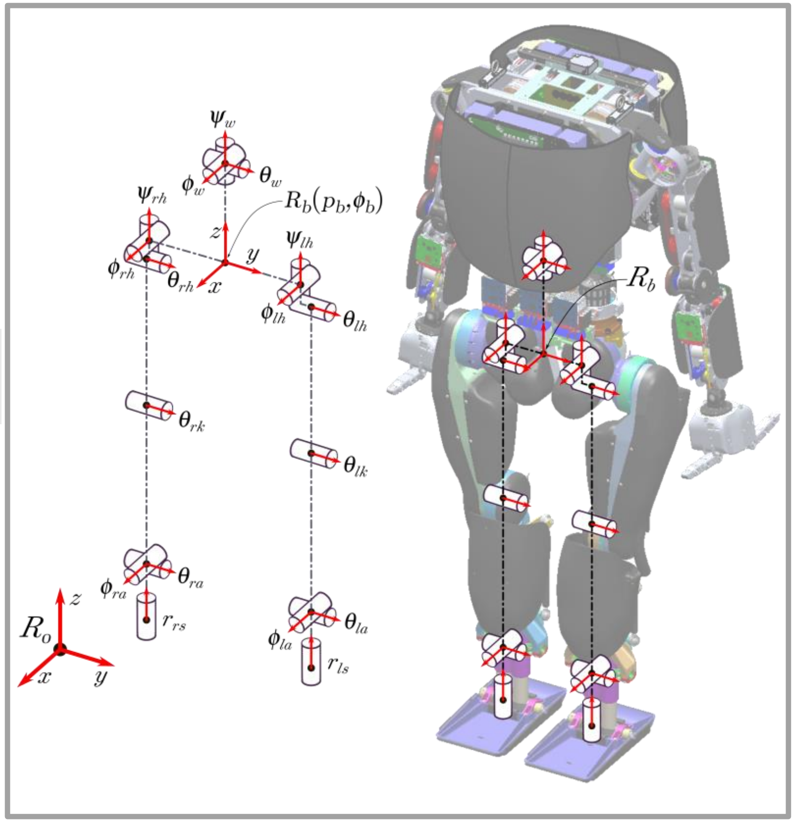
Assumptions

$\mathbf{f}$ locally Lipschitz continuous

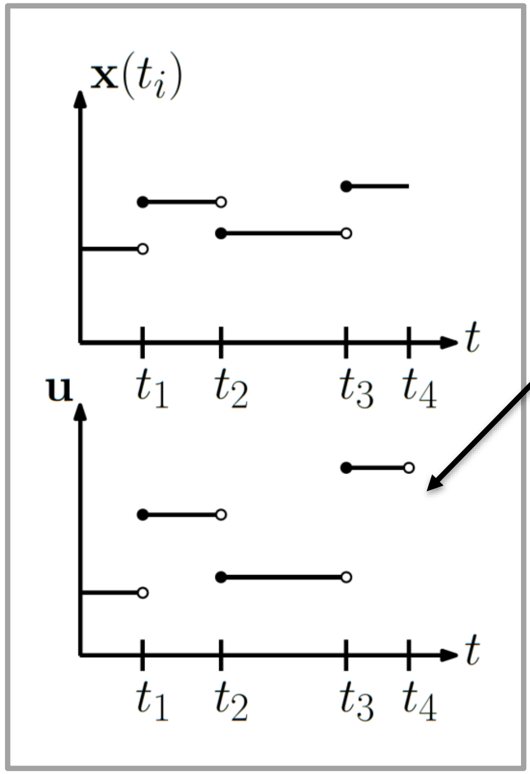
$$\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$$

Mathematical Model



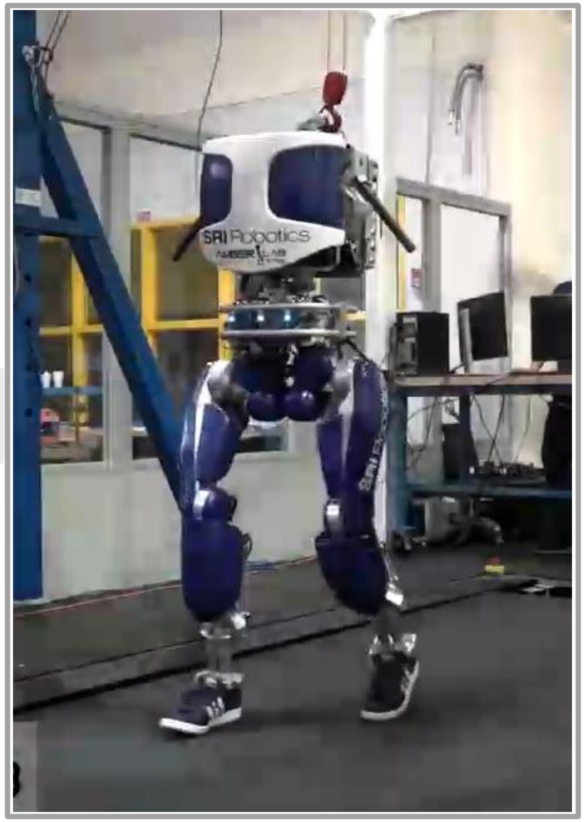
System Model

Event Triggered Control



Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \ \forall t \in [t_i, t_{i+1})$$



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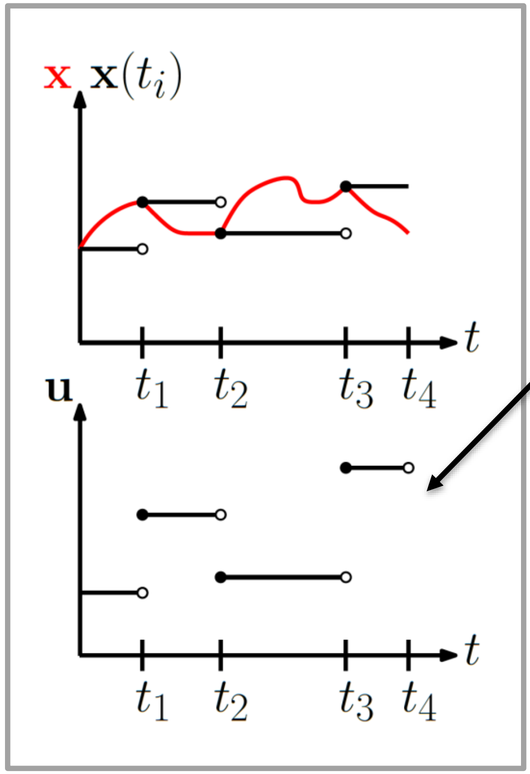
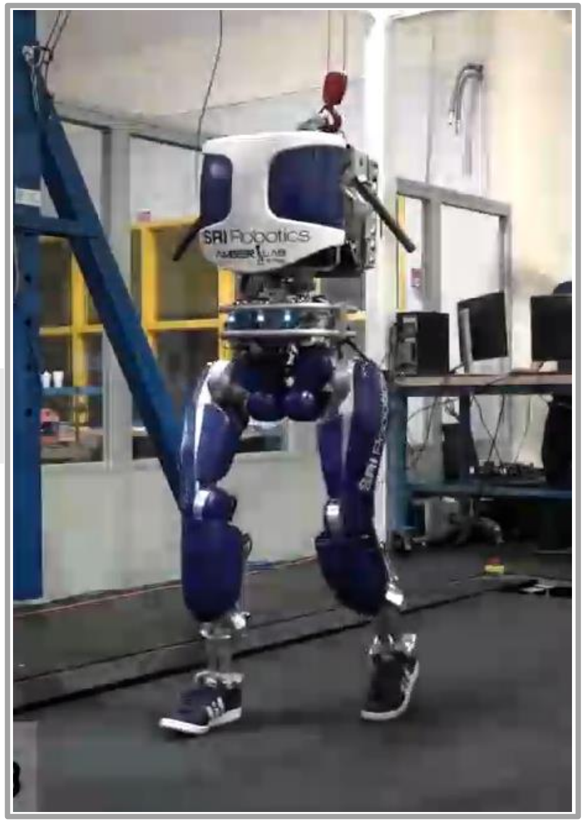


Sample-and-Hold

$$\mathbf{u}(t) = \mathbf{k}(\mathbf{x}(t_i)) \ \forall t \in [t_i, t_{i+1})$$

State Error

$$\mathbf{e}(t) = \mathbf{x}(t_i) - \mathbf{x}(t) \quad \forall t \in [t_i, t_{i+1})$$



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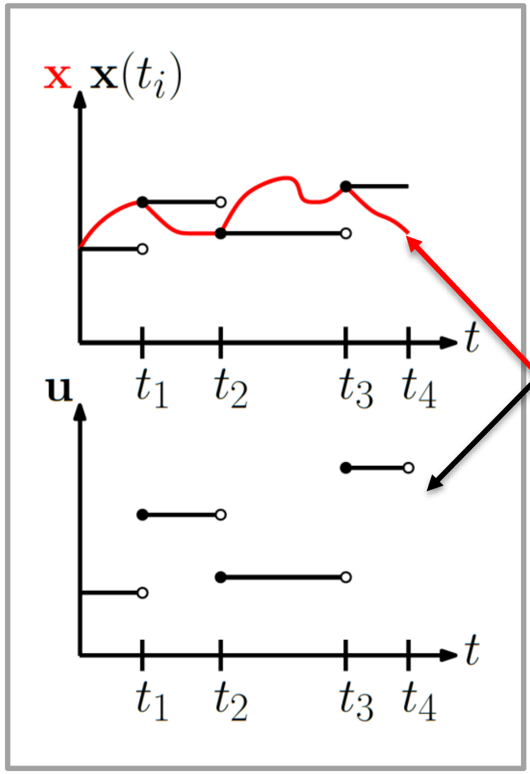

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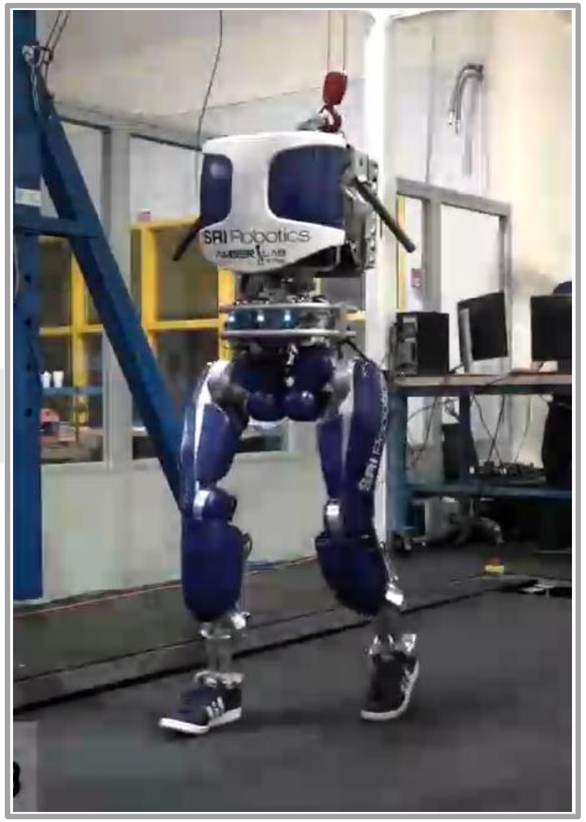

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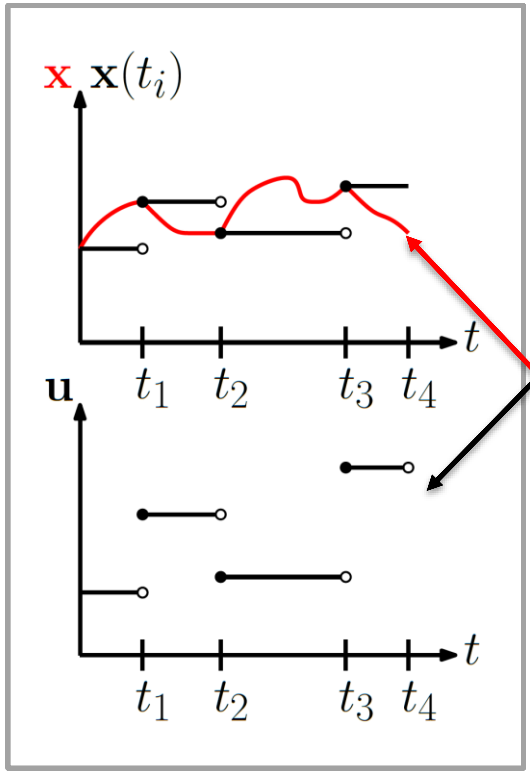

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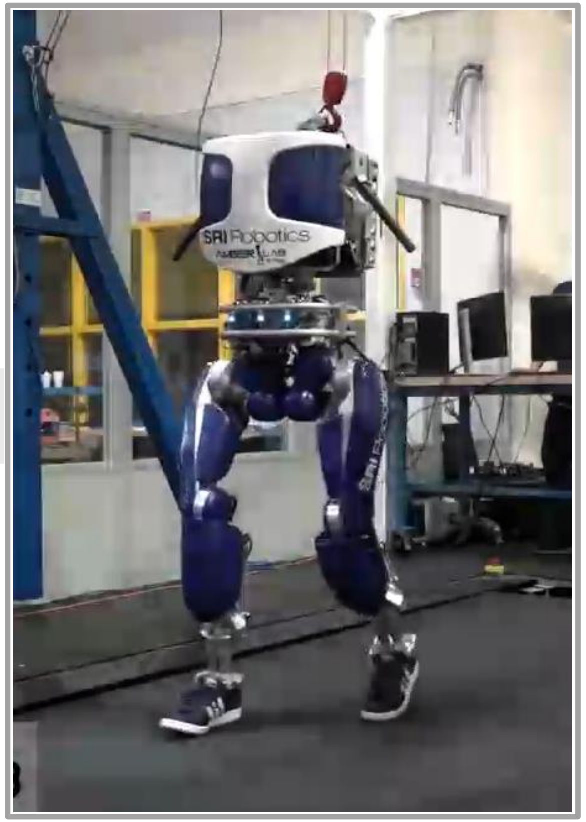
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x} + \mathbf{e}))$$



Input-to-State Stable Lyapunov Functions (ISS-LFs)

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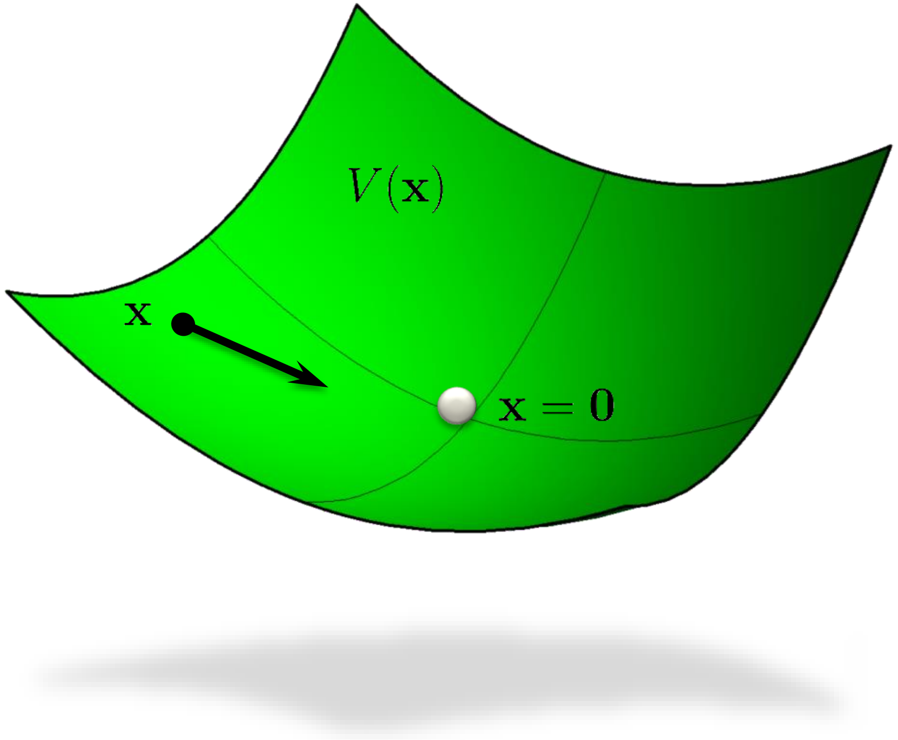
ISS-Lyapunov Function [2]

$$\alpha_1\left(\|\mathbf{x}\|_2\right) \leq V(\mathbf{x}) \leq \alpha_2\left(\|\mathbf{x}\|_2\right)$$

$$\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}+\mathbf{e}) \leq -\alpha_3(\|\mathbf{x}\|_2) + \gamma(\|\mathbf{e}\|_2)$$



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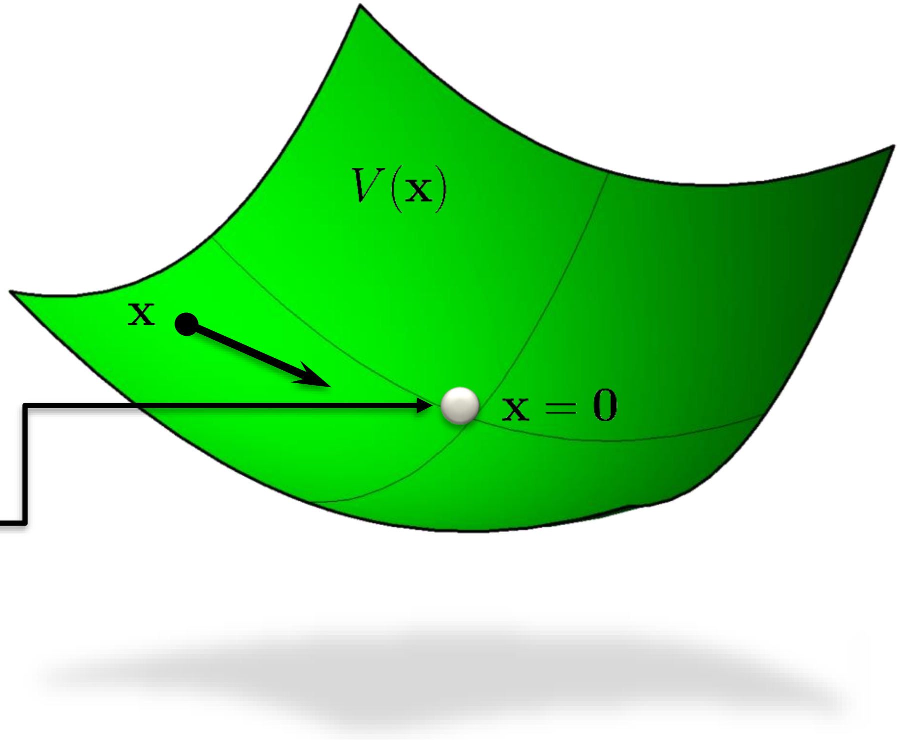

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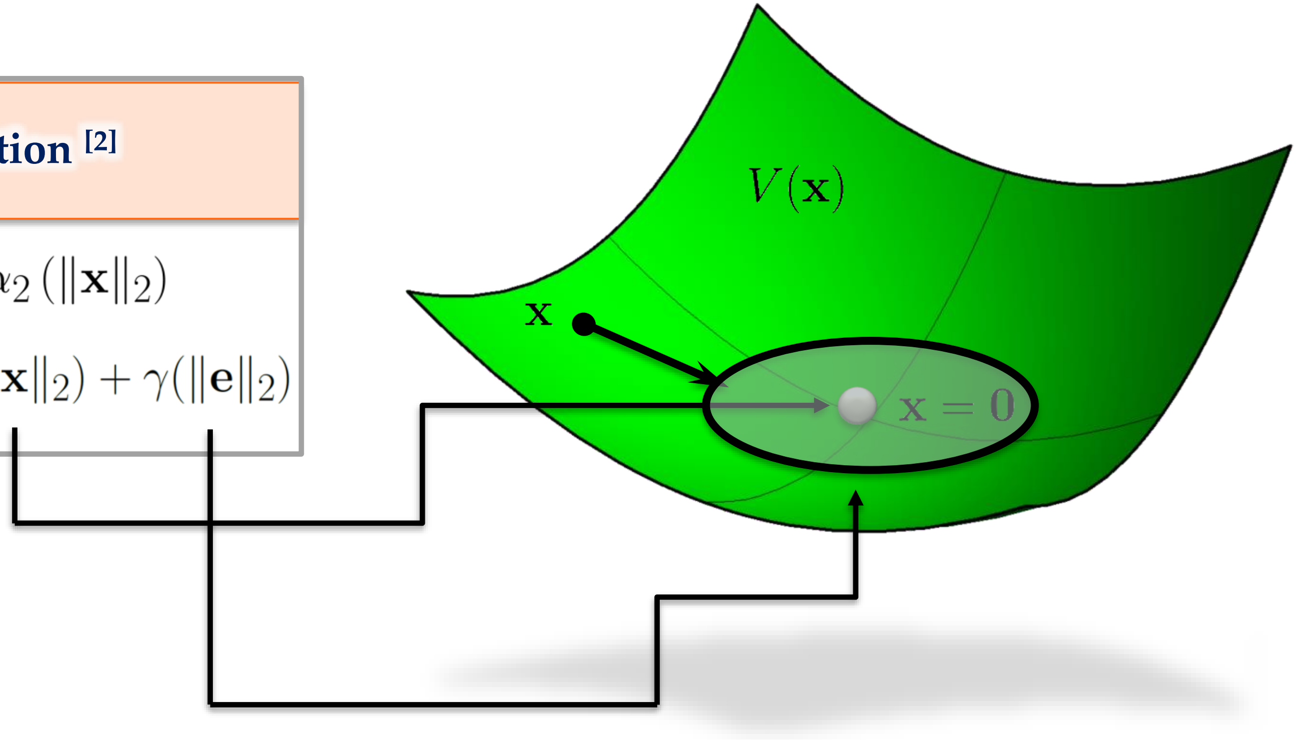

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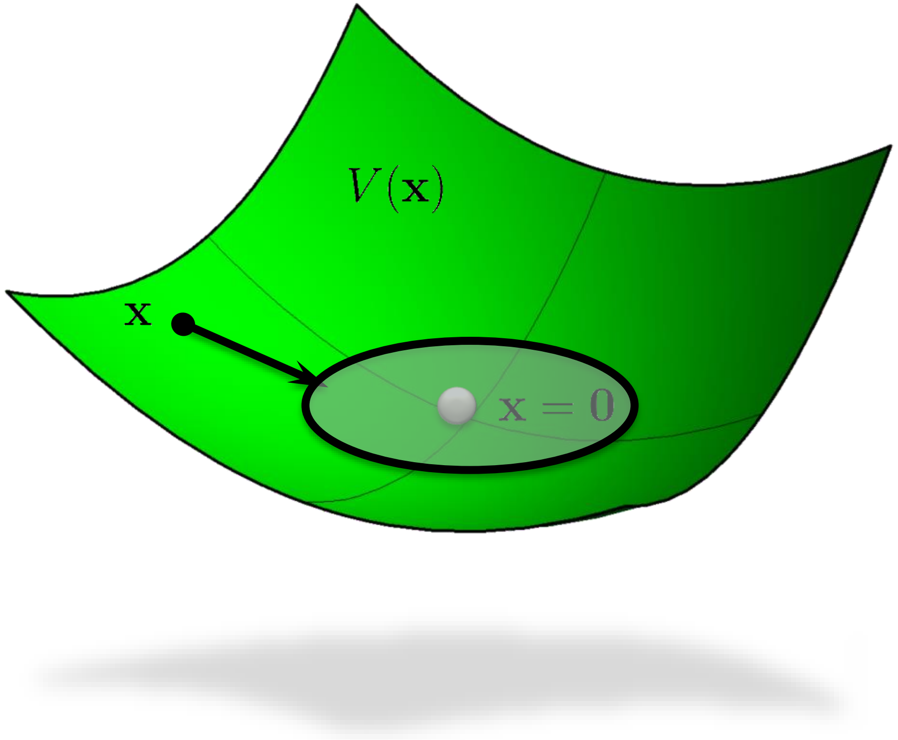
Input-to-State Stable Lyapunov Functions (ISS-LFs)

Stability Condition

$$\gamma(\|\mathbf{e}(t)\|_2) \leq \sigma\alpha_3(\|\mathbf{x}\|_2)$$

$$0 < \sigma < 1$$



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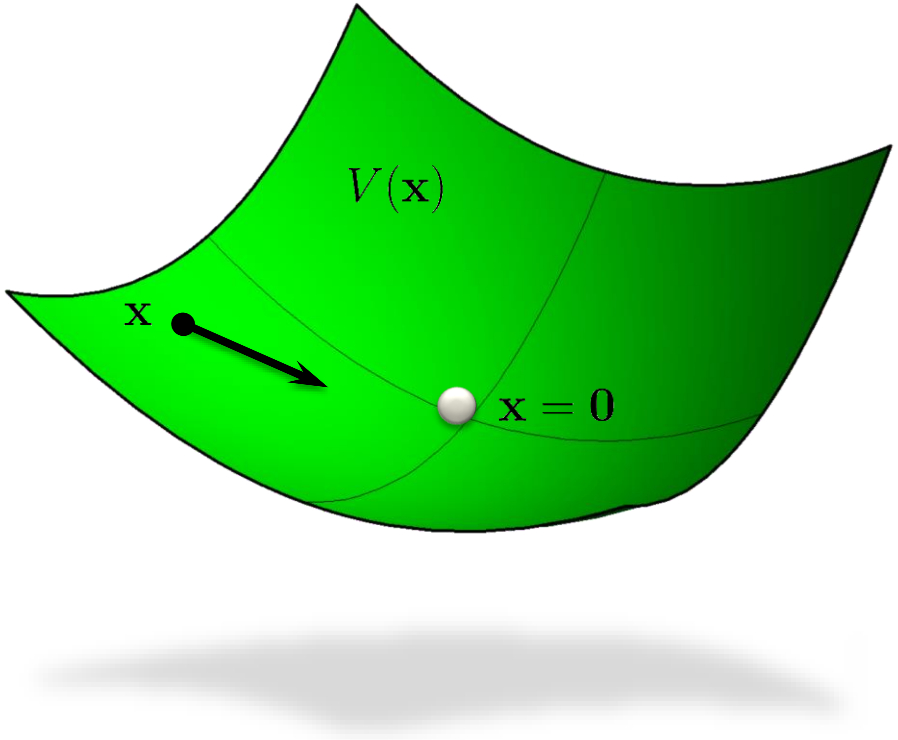

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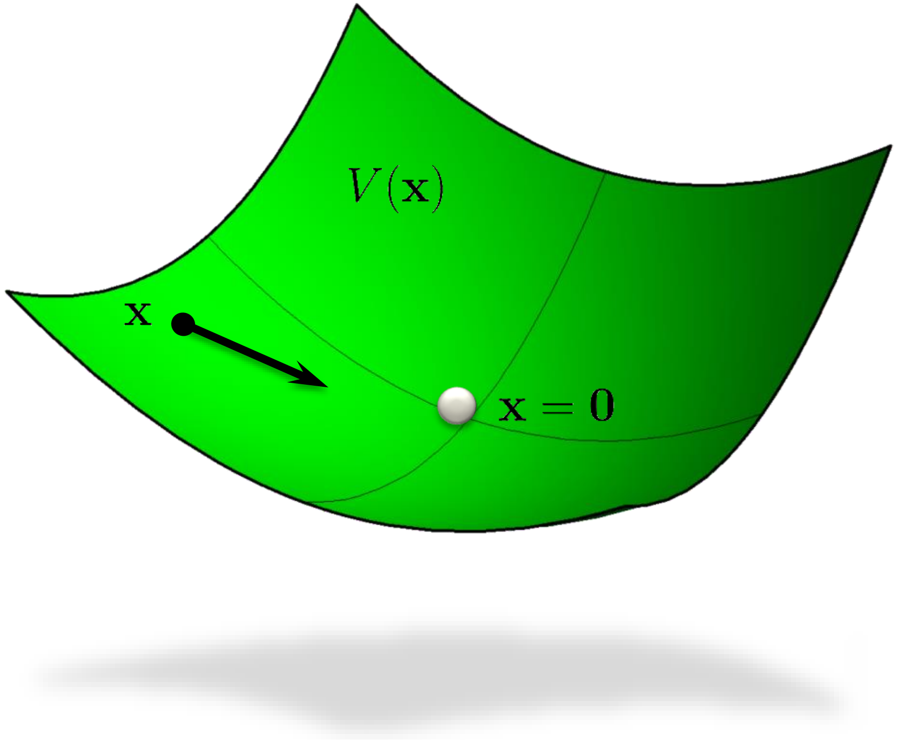

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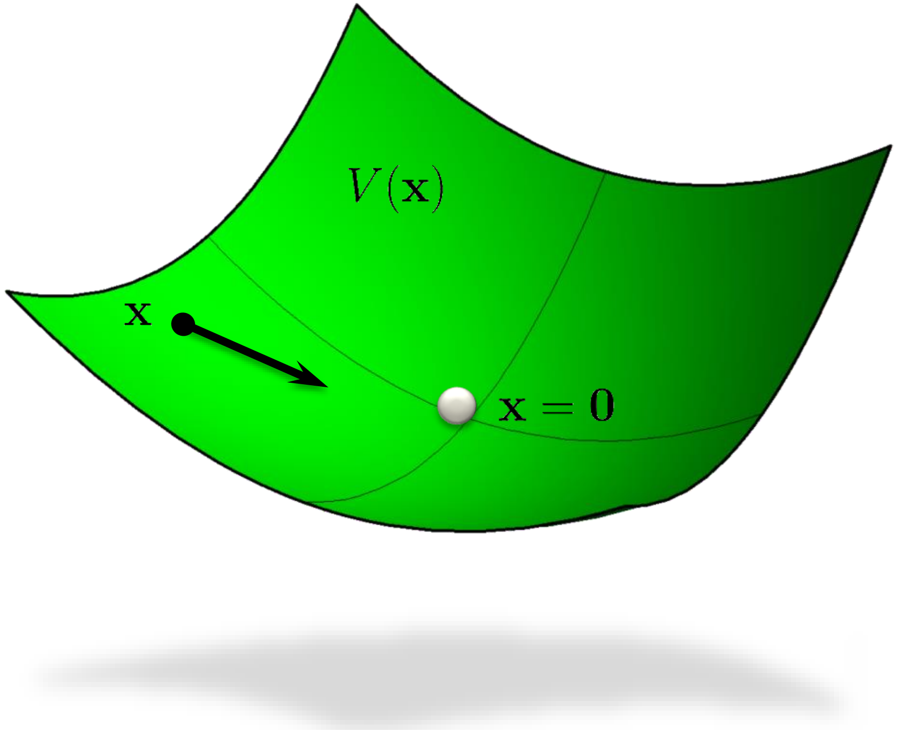
Trigger Law
$t_{i+1} = \min\{t \geq t_i \mid \gamma(\\|\mathbf{e}(t)\\|_2) = \sigma\alpha_3(\\|\mathbf{x}(t)\\|)\}$
MIET
$\mathbf{f},\ \mathbf{k},\ \alpha,\ \gamma$ Lipschitz on compacts



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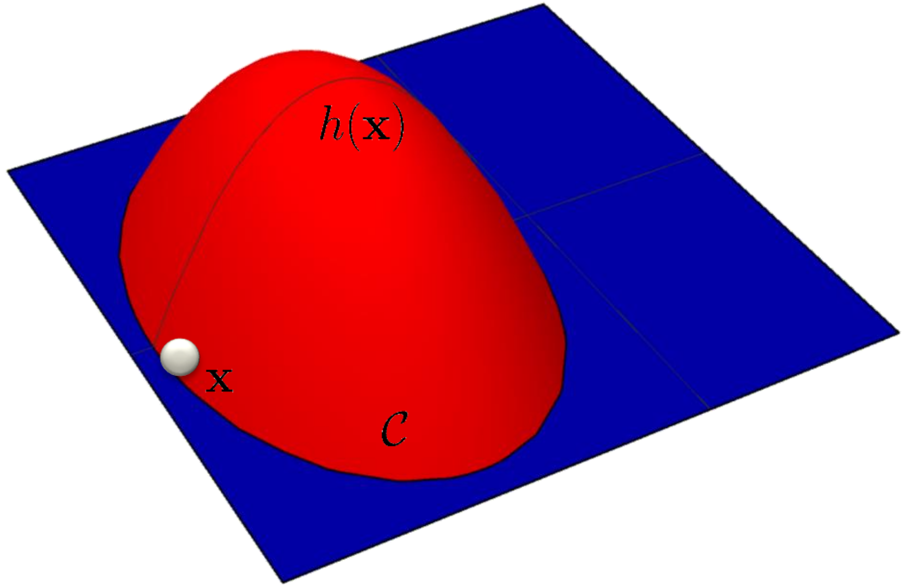
Input-to-State Safe Barrier Functions (ISSf-BFs)

Barrier Functions

$$\mathcal{C} \triangleq \{\mathbf{x} \in \mathbb{R}^n \ : \ h(\mathbf{x}) \geq 0\}$$



Input-to-State Safe Barrier Functions (ISSf-BFs)

Safety = Forward Invariance

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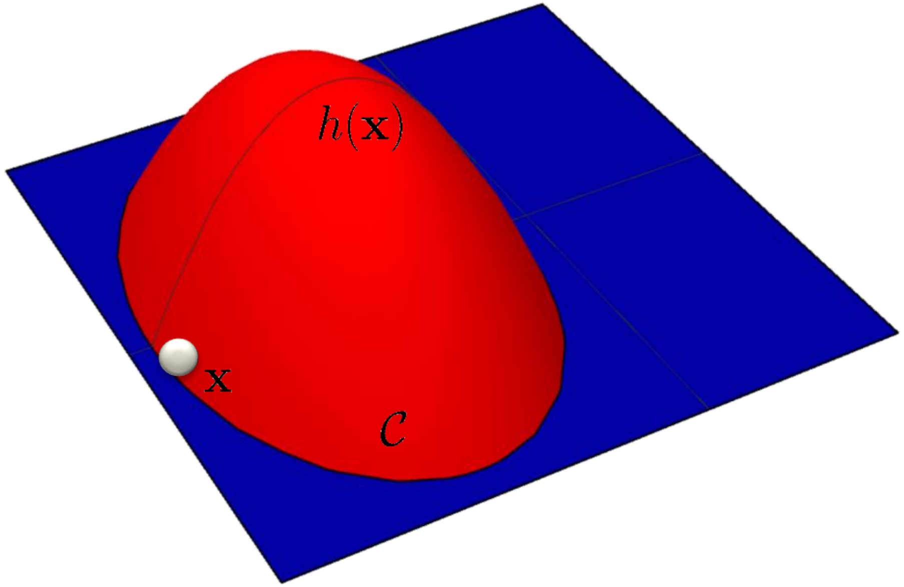

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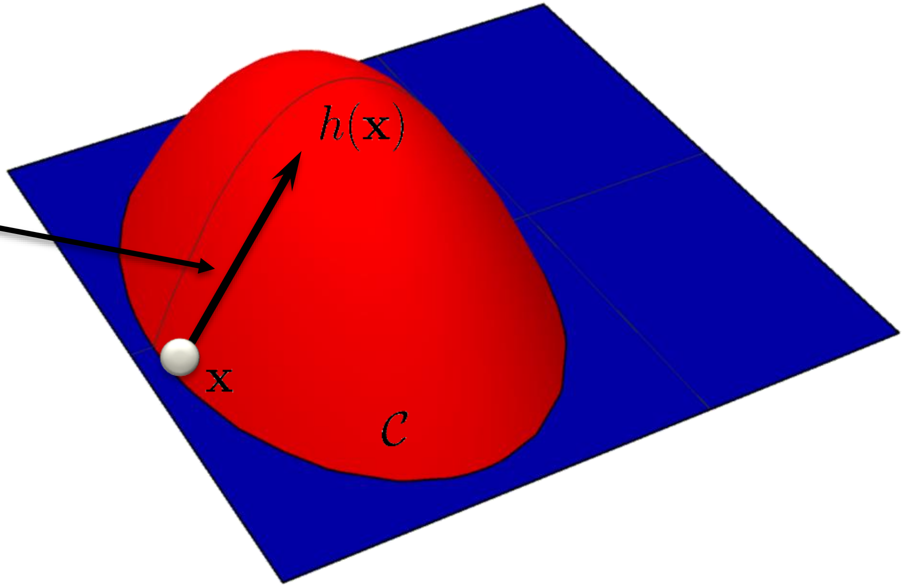

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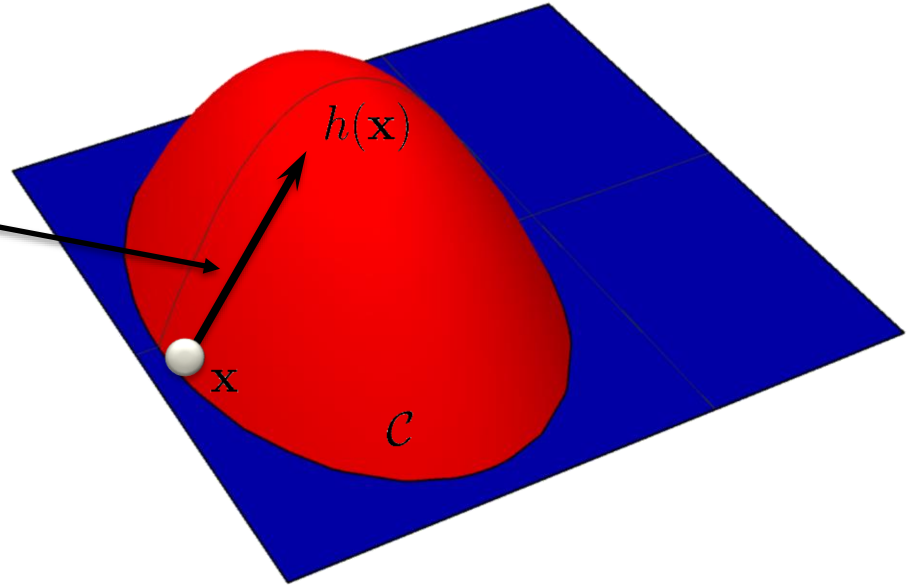
Input-to-State Safety (ISSf)[4]

$$\mathcal{C}_e \triangleq \{\mathbf{x} \in \mathbb{R}^n \ : \ h(\mathbf{x}) + \gamma(\|\mathbf{e}\|_2) \geq 0\}$$



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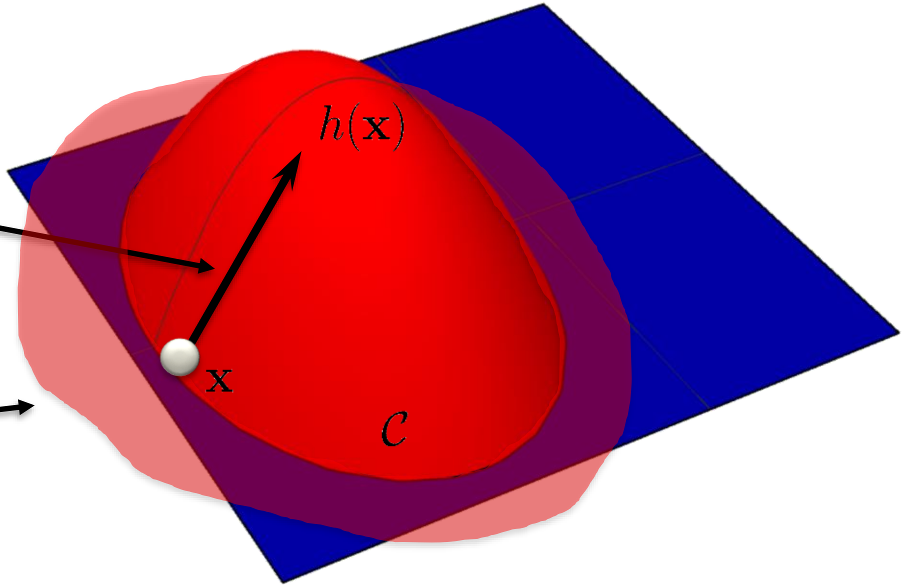

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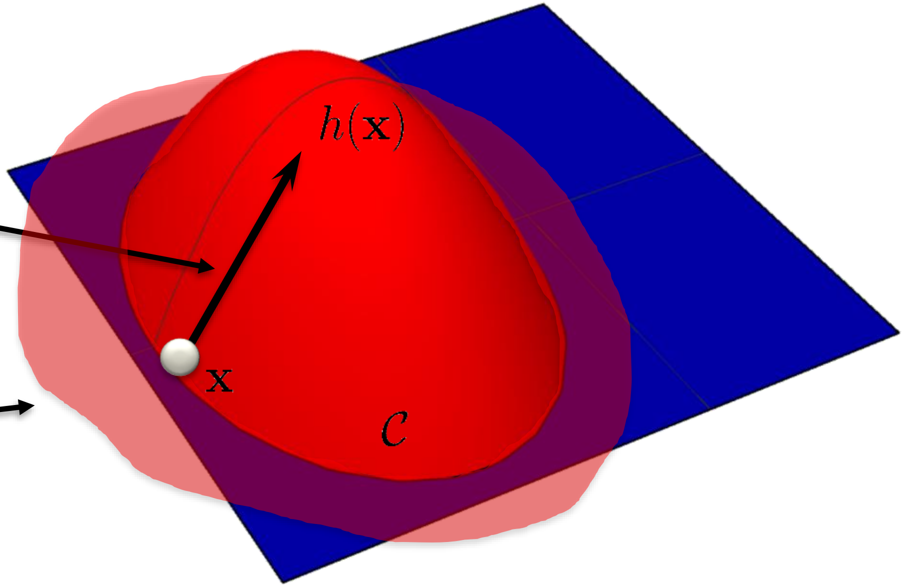

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Event Triggered Safety

Zeroing Barrier Functions

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[6] G. Yang, et. al., Self-triggered control for safety critical systems using control barrier functions, 2019.

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Lyapunov Barrier Functions

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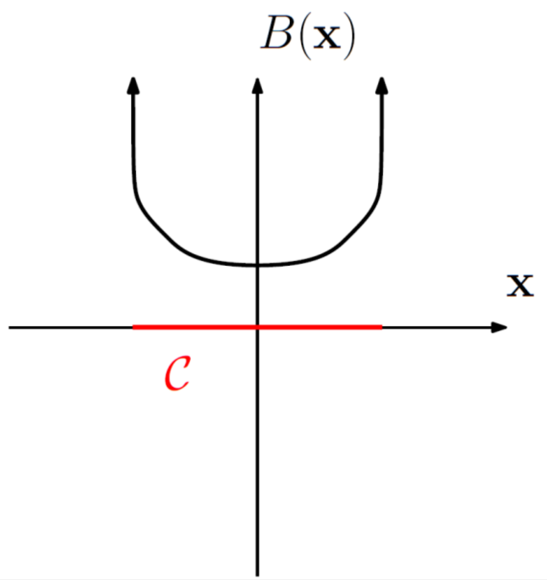

From Lyapunov to Barriers

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Safety Condition

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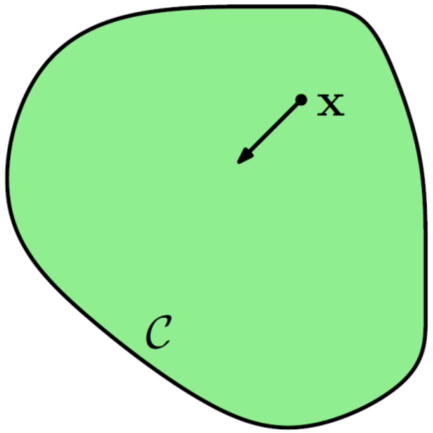
$$0 < \sigma$$



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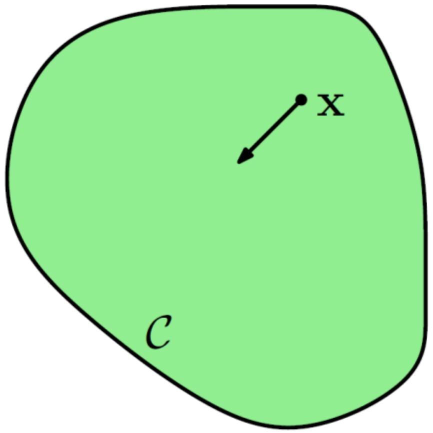

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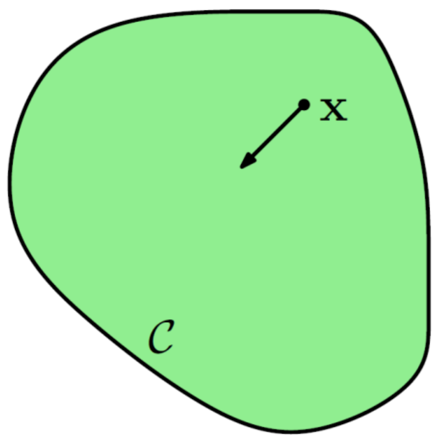

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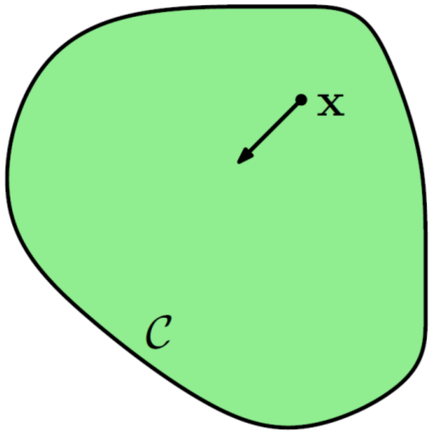
Outside Safe Set



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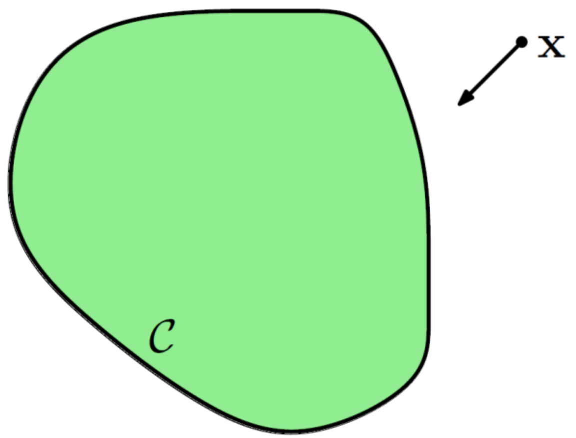
Outside Safe Set
$\mathbf{x} \notin \mathcal{C} \implies \alpha(h(\mathbf{x})) < 0$



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Outside Safe Set

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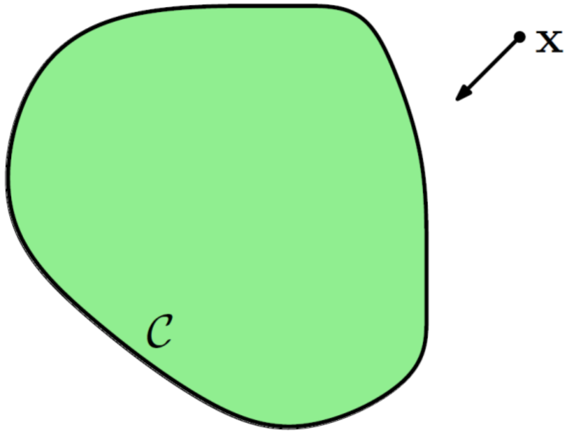
$$0 < \sigma < 1$$



From Lyapunov to Barriers

Safety Condition

$$\iota(\|\mathbf{e}(t)\|_2) \leq \sigma\alpha(h(\mathbf{x}(t)))$$

$$0 < \sigma$$

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}+\mathbf{e})) \geq -(1+\sigma)\alpha(h(\mathbf{x}))$$

Trigger Law

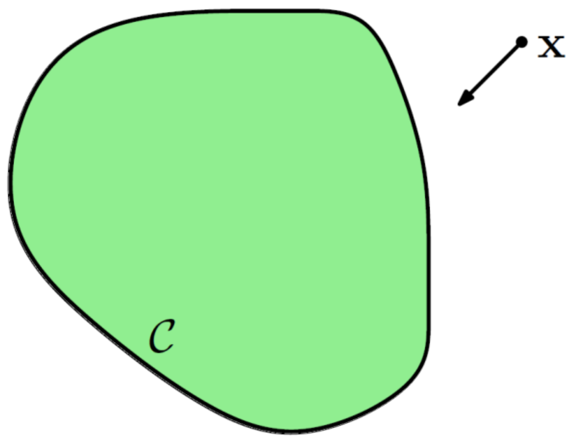
$$t_{i+1} = \min\{t \geq t_i \mid \iota(\|\mathbf{e}(t)\|_2) = \sigma\alpha(h(\mathbf{x}(t)))\}$$

Outside Safe Set

$$\iota(\|\mathbf{e}(t)\|_2) \leq \sigma|\alpha(h(\mathbf{x}(t)))|$$

$$0 < \sigma < 1$$

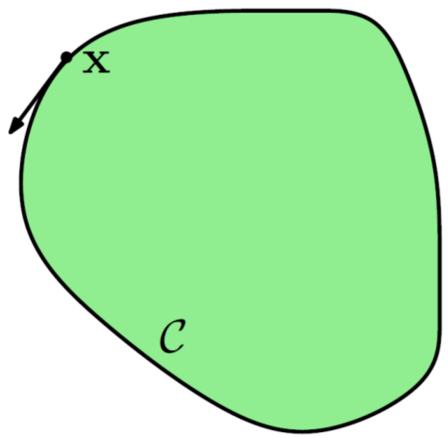
$$\mathbf{x} \notin \mathcal{C} \implies \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}+\mathbf{e})) \geq -(1-\sigma)\alpha(h(\mathbf{x}))$$



Minimum Interevent Time (MIET)?

Minimum Interevent Time (MIET)?

What about the boundary?



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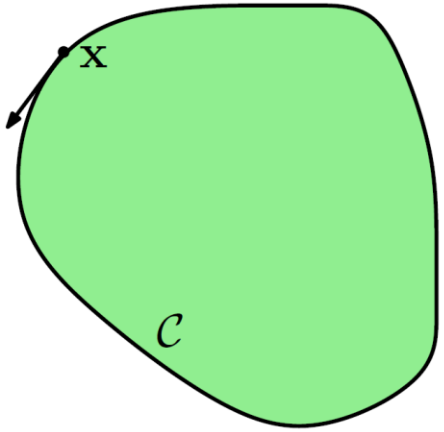

Minimum Interevent Time (MIET)?

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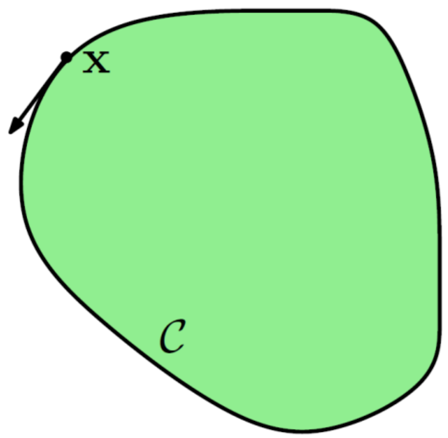
What about the boundary?

Trigger Law
$t_{i+1} = \min\{t \geq t_i \mid \iota(\lVert\mathbf{e}(t)\rVert_2) = \sigma\lvert\alpha(h(\mathbf{x}(t)))\rvert\}$
Tangential Motion
$\dot{\mathbf{e}}(t) \neq \mathbf{0}$ but $\dot{h}(\mathbf{x},\mathbf{e}) = \mathbf{0}$



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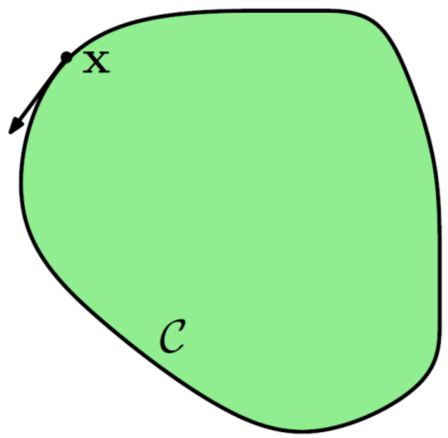
Stabilization is to an equilibrium point!



Example of MIET Failure

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System

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} u$$

Example of MIET Failure

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$$h(\mathbf{x}) = 1 - x_1^2 - x_2^2$$

Example of MIET Failure

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System
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Example of MIET Failure

Caltech UC San Diego

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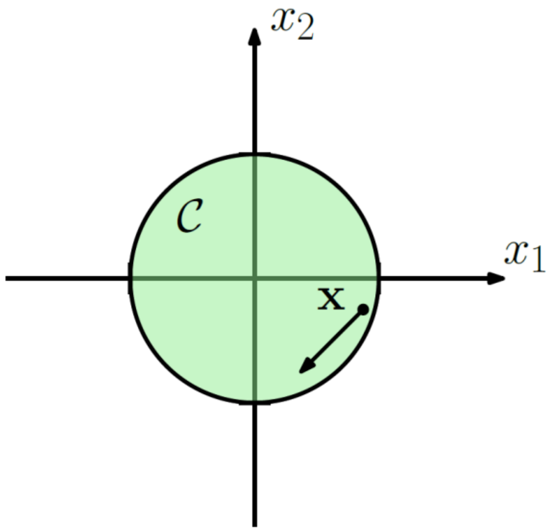

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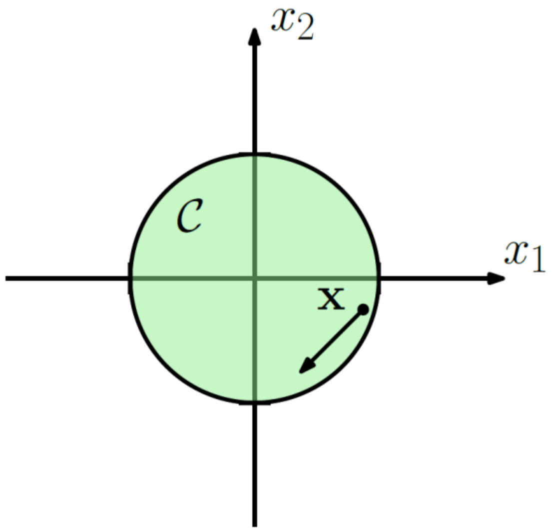
Event Triggered



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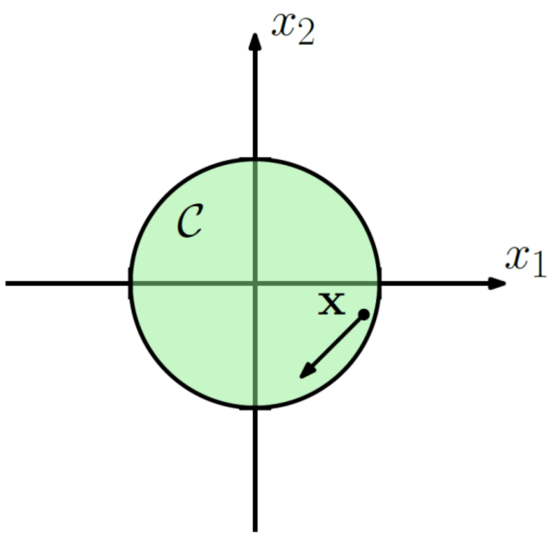

Event Triggered

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} k(\mathbf{x}(t_i)) & 1 \\ -1 & k(\mathbf{x}(t_i)) \end{bmatrix} \mathbf{x}(t)$$

Example of MIET Failure

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System

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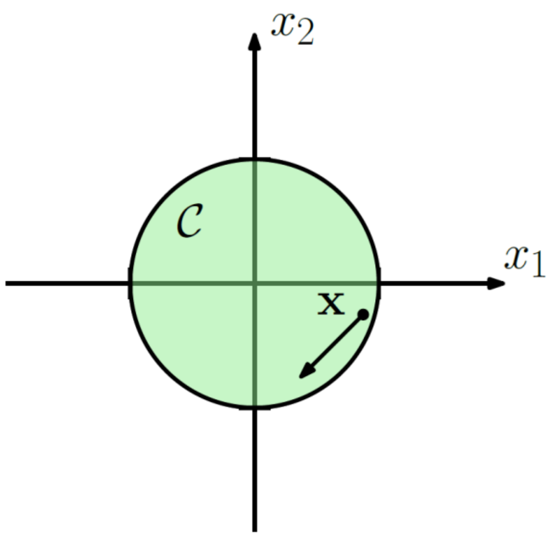

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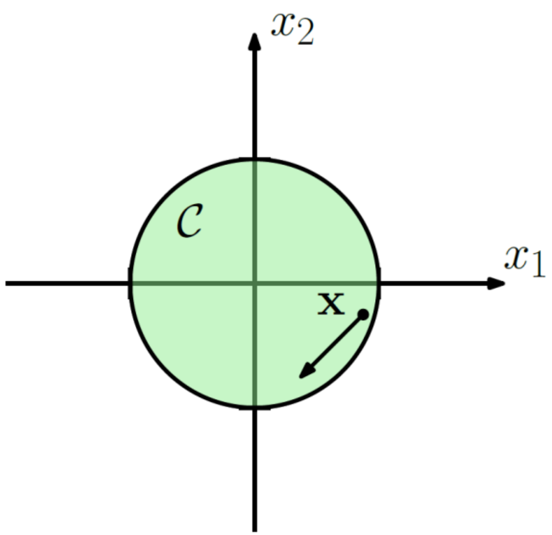

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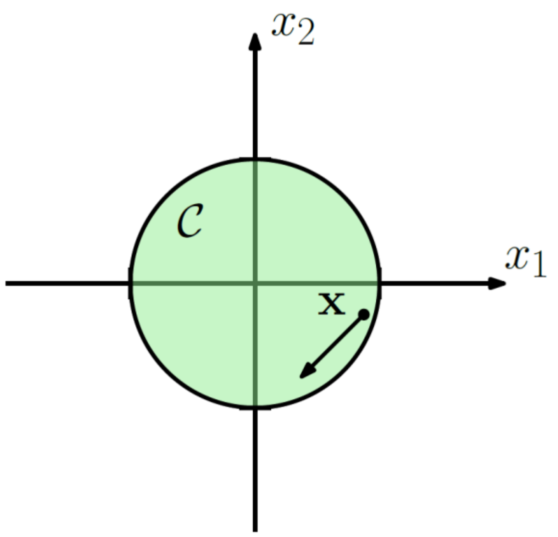

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Safety Condition

$$\dot{h}(\mathbf{x},\mathbf{e}) \geq \begin{cases} -(1+\sigma)h(\mathbf{x}), & \|\mathbf{x}\|_2 \leq 1 \\ -(1-\sigma)h(\mathbf{x}), & 1 < \|\mathbf{x}\|_2 \leq r \end{cases}$$

Proof of MIET Failure

Assume MIET
$t_{i+1} \geq t_i + \tau$

Proof of MIET Failure

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Infinite Events

$$\forall\, T > 0, \exists\, t_i > T$$

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$\lim_{t_i \to \infty} h(\mathbf{x}(t_i)) = 0$

Proof of MIET Failure

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$$t_i^\star < t_i + \tau$$
$$2r^3 \|\mathbf{e}(t_i^\star)\|_2 \geq \sigma |h(\mathbf{x}(t_i^\star))|$$

Proof of MIET Failure

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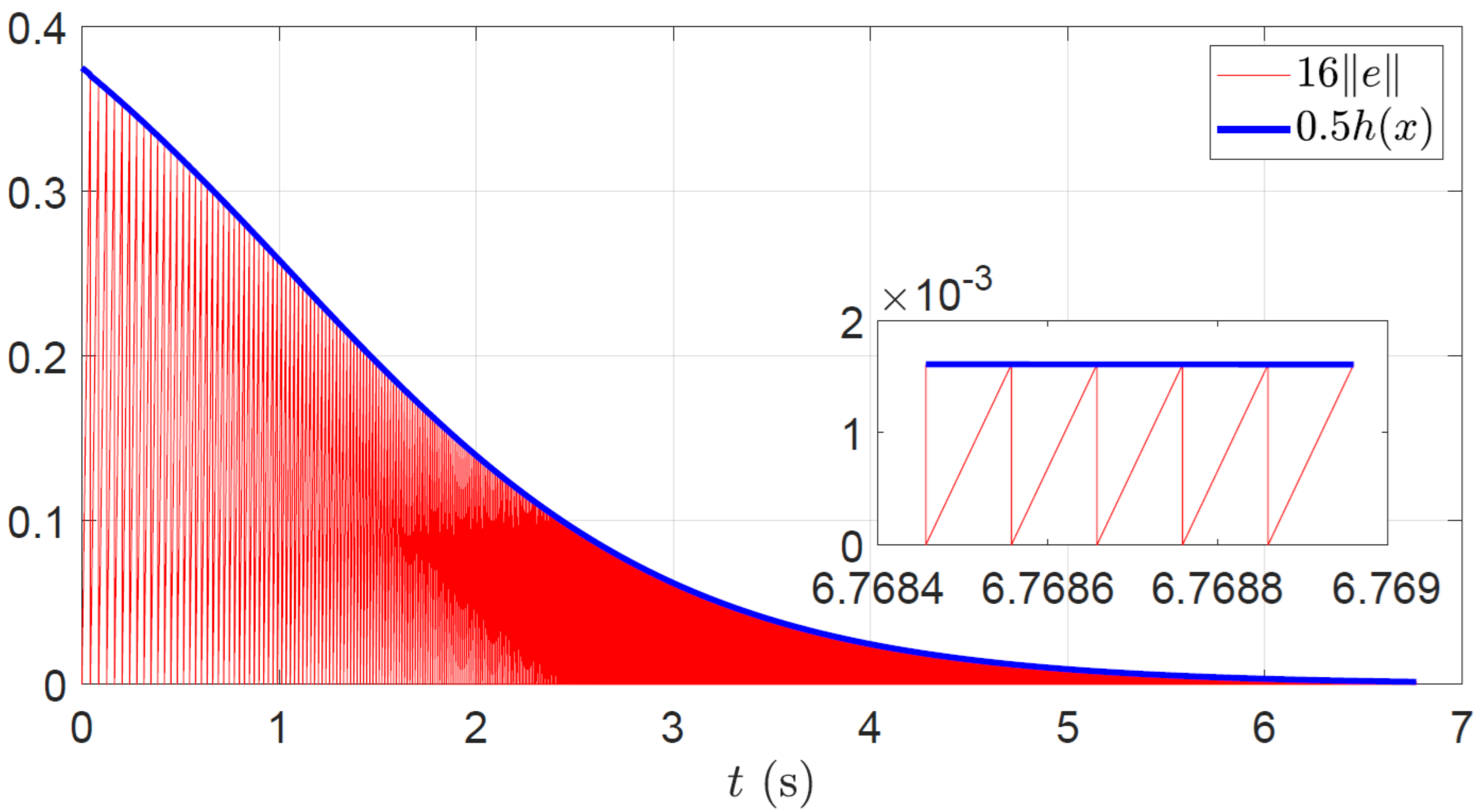
Tangential dynamics on the boundary! $\dot{\mathbf{e}}(t_i) \neq \mathbf{0}$

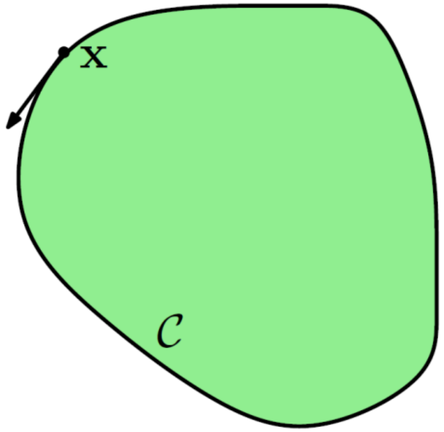
[9] D. P. Borgers, W. P. M. H. Heemels, Event-Separation Properties of Event-Triggered Control Systems, 2014.

Understanding aCBF Conservativeness



Strong ISSf Barrier Property

Can we eliminate tangential motion on boundary?



Strong ISSf Barrier Property

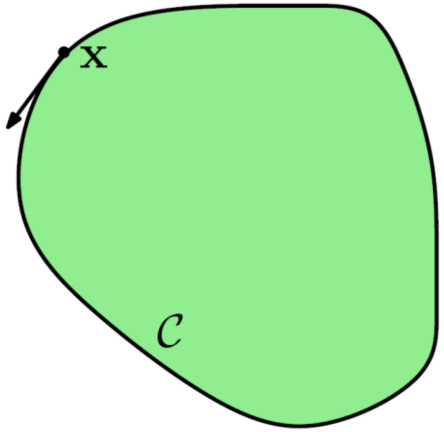
Caltech UC San Diego

Can we eliminate tangential motion on boundary?

Strong ISSF Barrier Property

Definition 6 (*Strong ISSf Barrier Property*)**.** An ISSf-BF h satisfies the *strong ISSf barrier property* if there exists $d \in \mathbb{R}$ with $d > 0$ such that for all $\mathbf{x}, \mathbf{e} \in \mathbb{R}^n$:

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}+\mathbf{e})) \geq -\alpha(h(\mathbf{x})) + d - \iota(\|\mathbf{e}\|_2),$$



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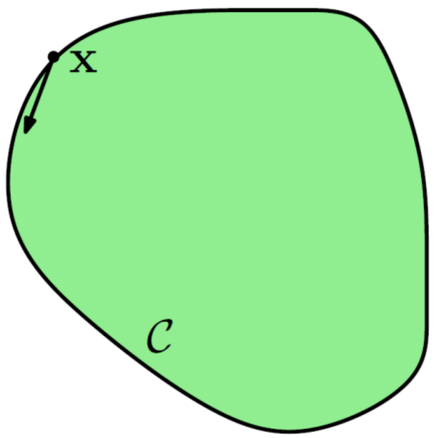

Event Triggered Safety with MIET

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Assumptions

Theorem 1 (Trigger Law for Safety Critical Systems)**.** *Let h be an ISSf-BF for* (6) *on a set $\mathcal{C} \subset \mathbb{R}^n$ defined as in* (11a)-(11c), *with corresponding functions $\alpha \in \mathcal{K}_{\infty,e}$ and $\iota \in \mathcal{K}_\infty$. Let $\beta \in \mathcal{K}_{\infty,e}$, $\sigma \in (0,1]$. If the following assumptions hold:*

1) *h satisfies the strong ISSf barrier property for a constant $d \in \mathbb{R}$, $d > 0$,*
2) *ι is Lipschitz continuous with Lipschitz constant L_ι,*
3) *there exists $F \in \mathbb{R}$, $F > 0$, such that for all $\mathbf{x}, \mathbf{e} \in \mathbb{R}^n$:*

$$\|\mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}+\mathbf{e}))\|_2 \leq F,$$

4) *$\beta(r) \geq \alpha(r)$ for all $r \in \mathbb{R}$,*

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Trigger Law

$$t_{i+1} = \min \{t \geq t_i \mid \iota(\|\mathbf{e}(t)\|_2) = \beta(h(\mathbf{x}(t))) - \alpha(h(\mathbf{x}(t))) + \sigma d\}$$

Event Triggered Safety with MIET

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Safe + MIET

$$\dot{h}(\mathbf{x}, \mathbf{e}) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}+\mathbf{e})) \geq -\beta(h(\mathbf{x}))$$

$$t_{i+1} - t_i \geq \tau \triangleq \frac{\sigma d}{L_\iota F}, \quad i \in \mathbb{N}$$

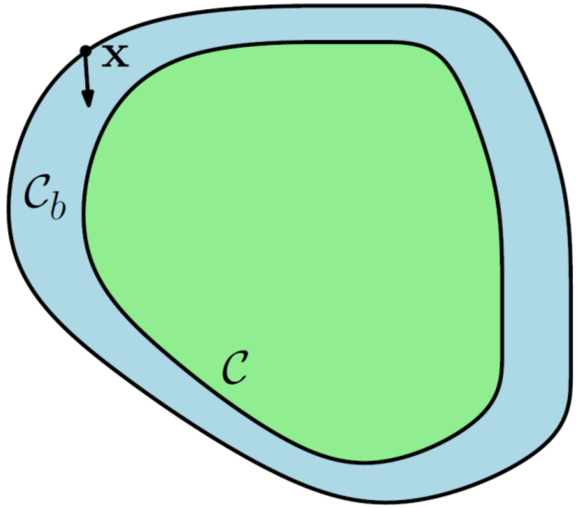
Strong Barrier Property via ISSf

Caltech UC San Diego

Extended Set

Theorem 2 (Strong ISSf Barrier Property in Supersets). *Let h be an ISSf-BF for (6) on a set $\mathcal{C} \subset \mathbb{R}^n$ defined as in (11a)-(11c), with corresponding functions $\alpha \in \mathcal{K}_{\infty,e}$ and $\iota \in \mathcal{K}_\infty$. Then the function h_b defined as $h_b(\mathbf{x}) = h(\mathbf{x}) + b$, with $b \in \mathbb{R}$, $b > 0$, is an ISSf-BF satisfying the strong ISSf barrier property on the set $\mathcal{C}_b$ defined as:*

$$\mathcal{C}_b \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid h_b(\mathbf{x}) \geq 0\} \tag{25}$$



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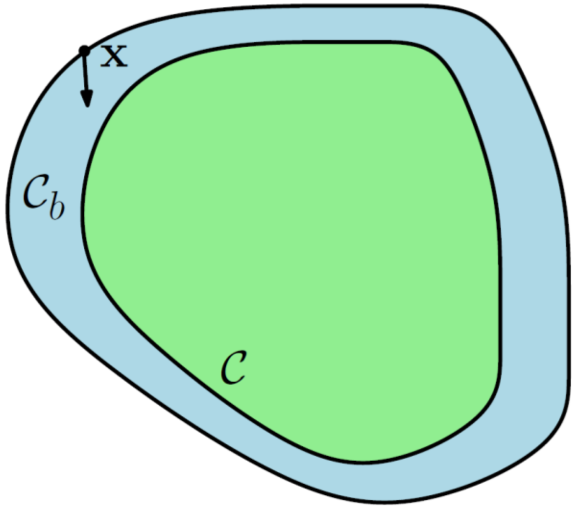
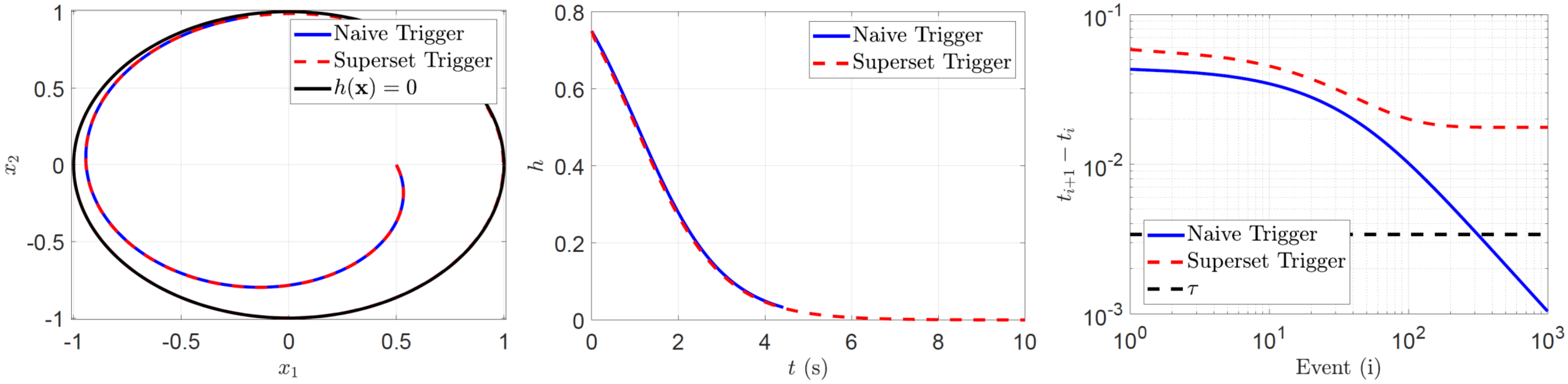


Set Dialation

Corollary 1 (Superset Trigger Law). *If h is an ISSf-BF for* (6) *on the set $\mathcal{C}$ satisfying Assumptions (2-4) of Theorem 1, then h_b is an ISSf-BF for* (6) *on the set $\mathcal{C}_b$ satisfying Assumptions (1-4) of Theorem 1 such that the corresponding trigger law renders $\mathcal{C}_b$ safe and asymptotically stable with a MIET.*

Simulation Results



Conclusions

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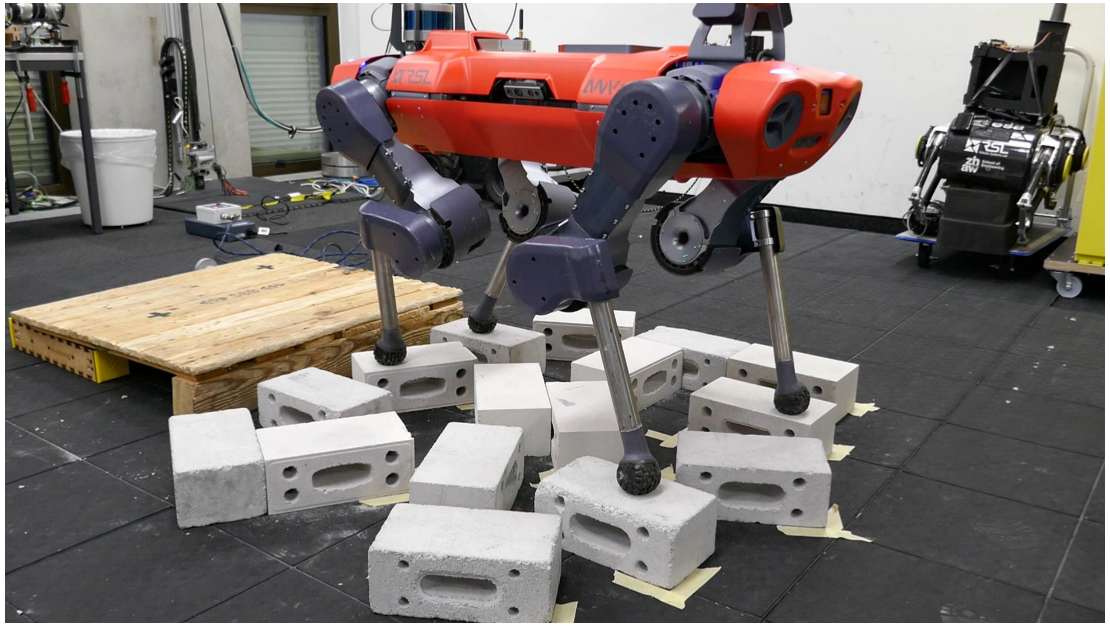
- **Input-to-State Safe Barrier Functions** offer solution for resource efficient event triggered safety

- Event-triggered set invariance faces challenges not encountered by event-triggered stabilization methods

- Event-triggered stabilization and safety with can be achieved simultaneously using multiple trigger laws.



Thank You!

Safety-Critical Event Triggered Control
via Input-to-State Safe Barrier Functions

Andrew Taylor Pio Ong Jorge Cortés Aaron Ames