Adaptive Safety with Control Barrier Functions

Andrew Taylor Aaron Ames

Computing and Mathematical Sciences
California Institute of Technology

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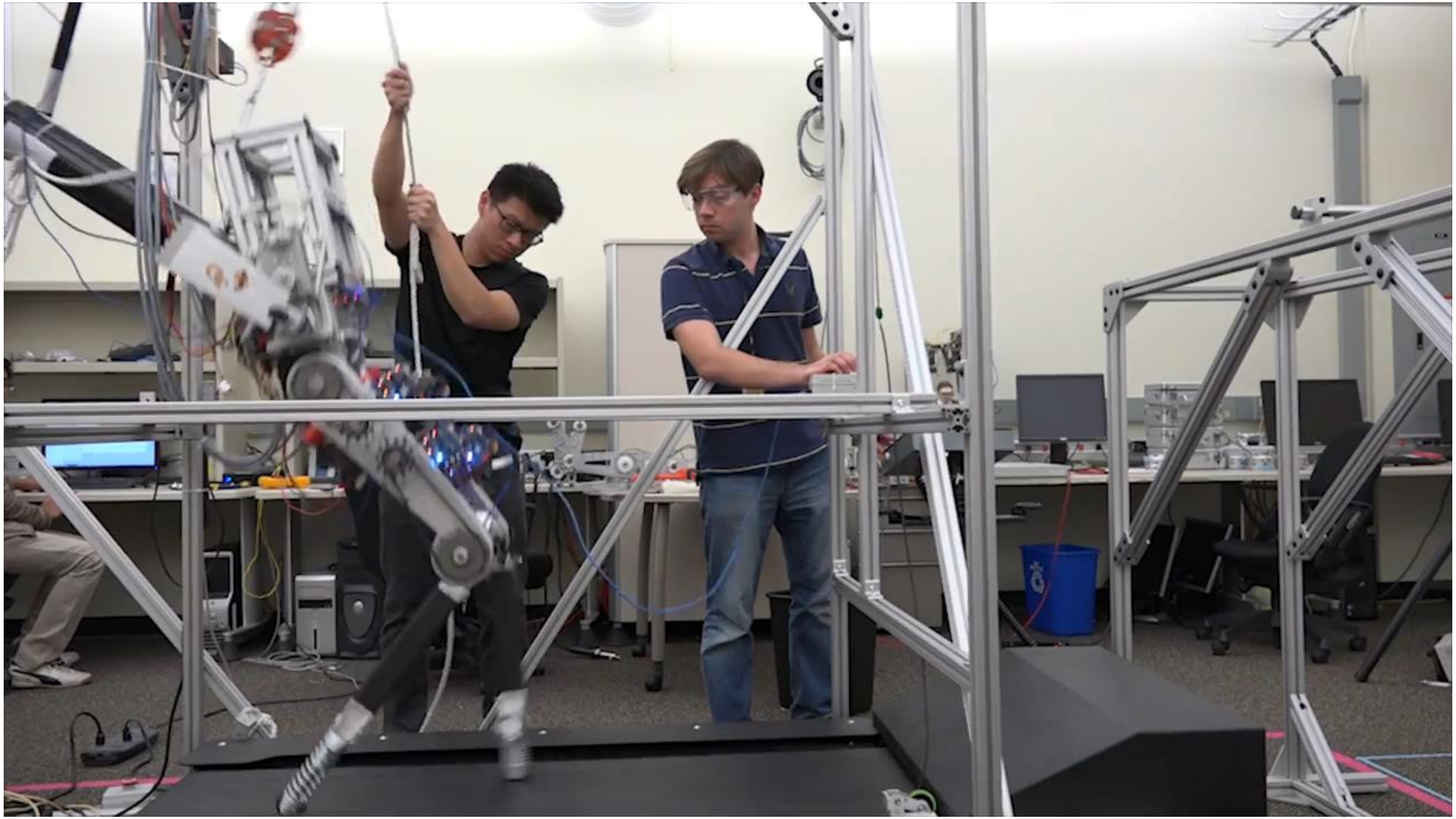
American Control Conference 2020

Control in the real world is hard

Caltech



But: Pretty when it works…

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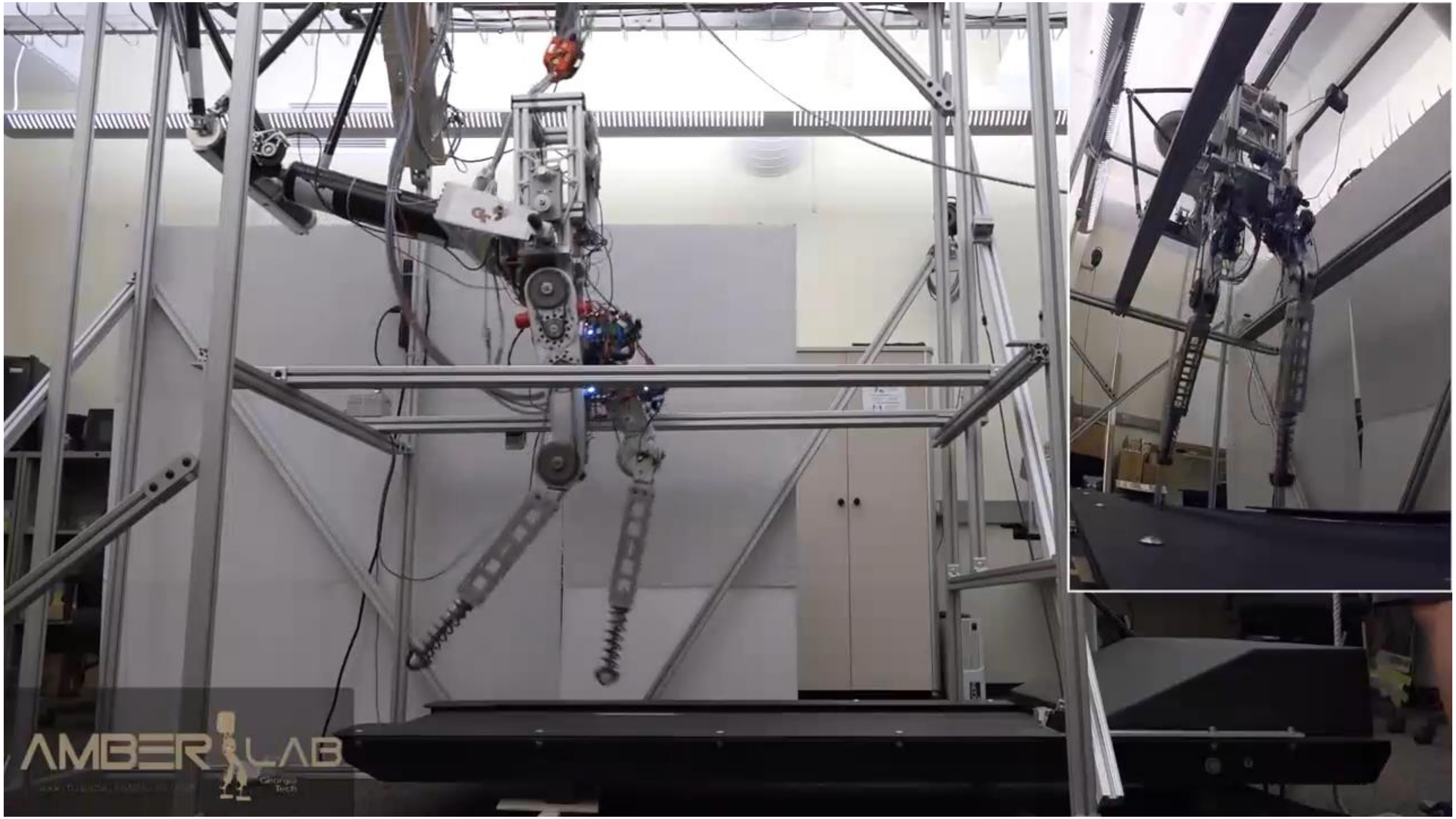
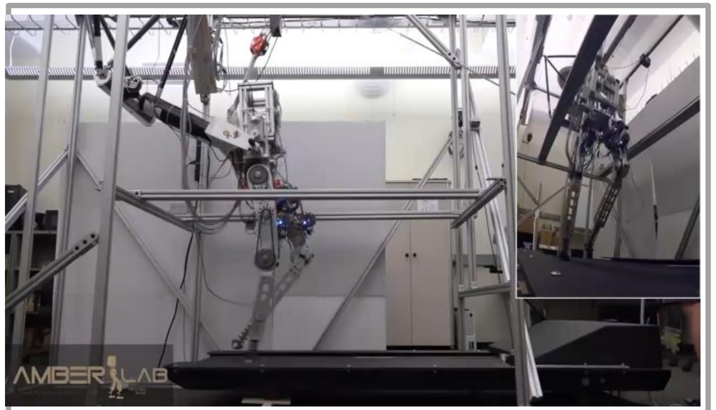


[1] W. Ma, et al., Bipedal robotic running with durus-2d: Bridging the gap between theory and experiment

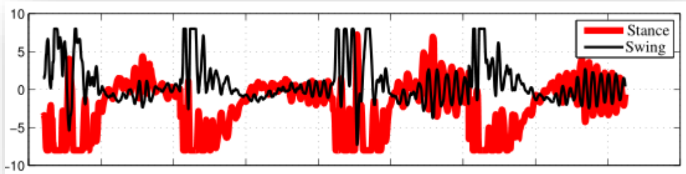
Claim: Need to Bridge the Gap

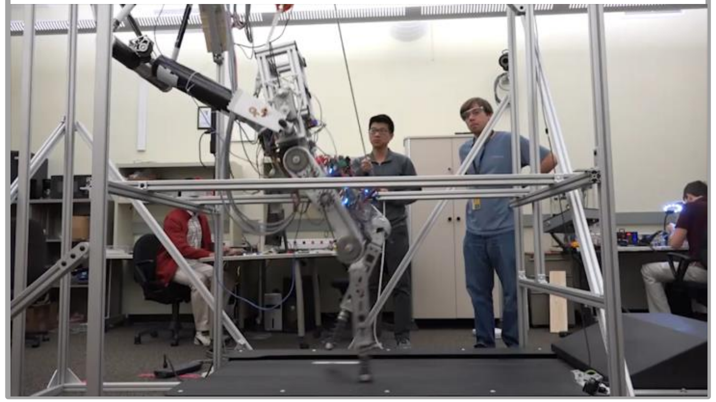


$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2^2$$
$$\text{s.t.} \quad \dot{h}(\mathbf{x},\mathbf{u}) \geq -\alpha(h(\mathbf{x}))$$

Theorems & Proofs

Bridge the Gap





Experimental Realization

Contributions

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- Framework for achieving safety of uncertain systems via **Adaptive Control Barrier Functions (aCBFs)**
- Analysis of the conservative nature of the aCBF conditions through a pathological example
- Combination of adaptive control methods for stabilization and safety demonstrated on an Adaptive Cruise Control system

System Dynamics

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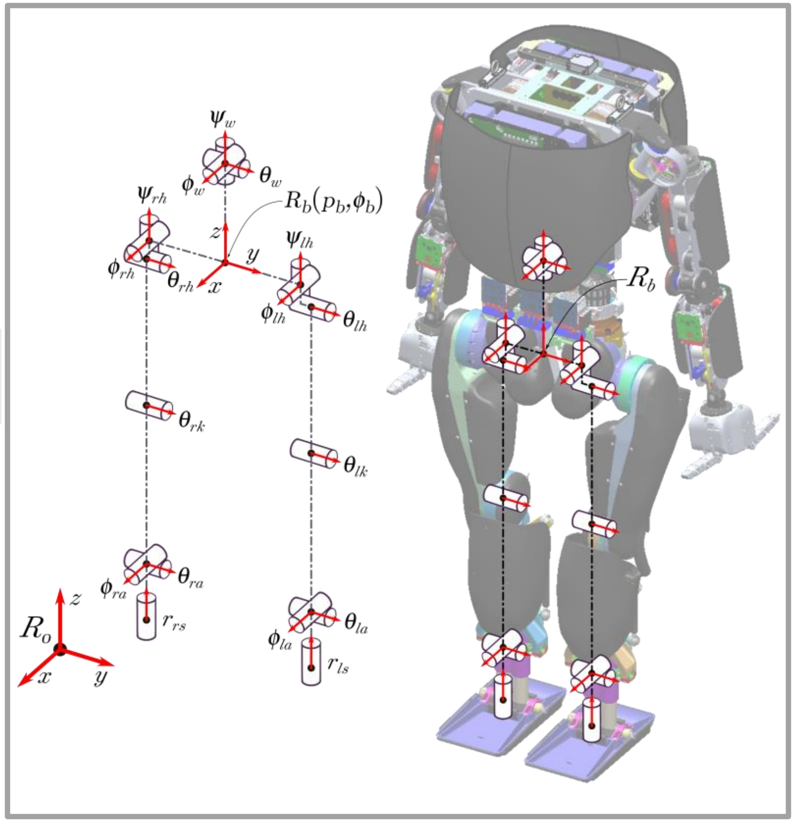
Equations of Motion

$$\widehat{\dot{\mathbf{x}}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \quad \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^{n \times m}$$

Mathematical Model



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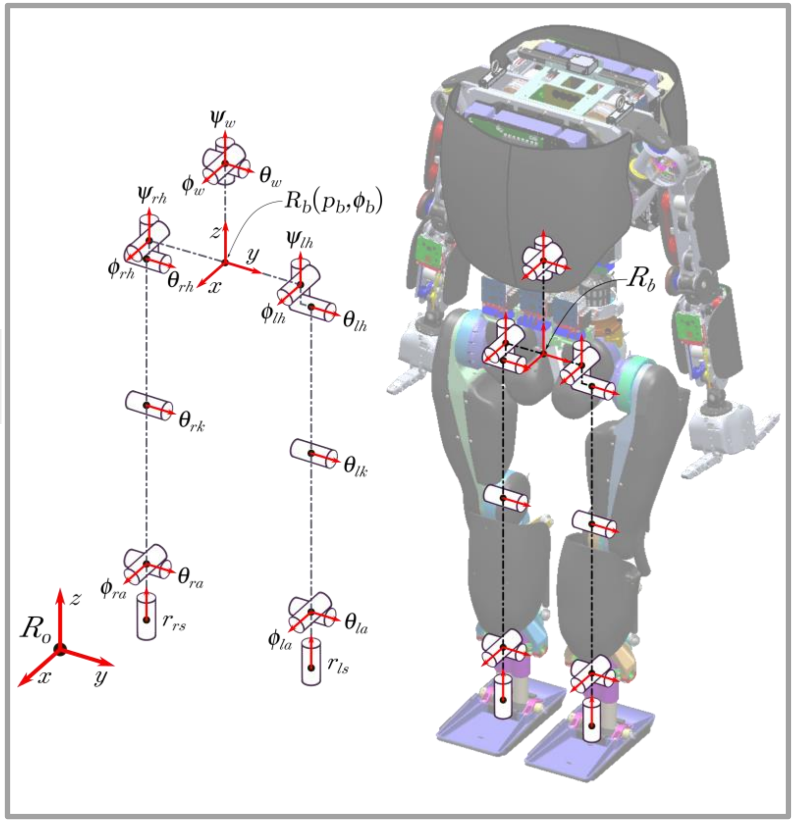
Assumptions

$\mathbf{f}, \mathbf{g}$ locally Lipschitz continuous

$$\mathbf{f}(\mathbf{0}) = \mathbf{0}$$

Mathematical Model



System Model

Control Lyapunov Functions (CLFs)

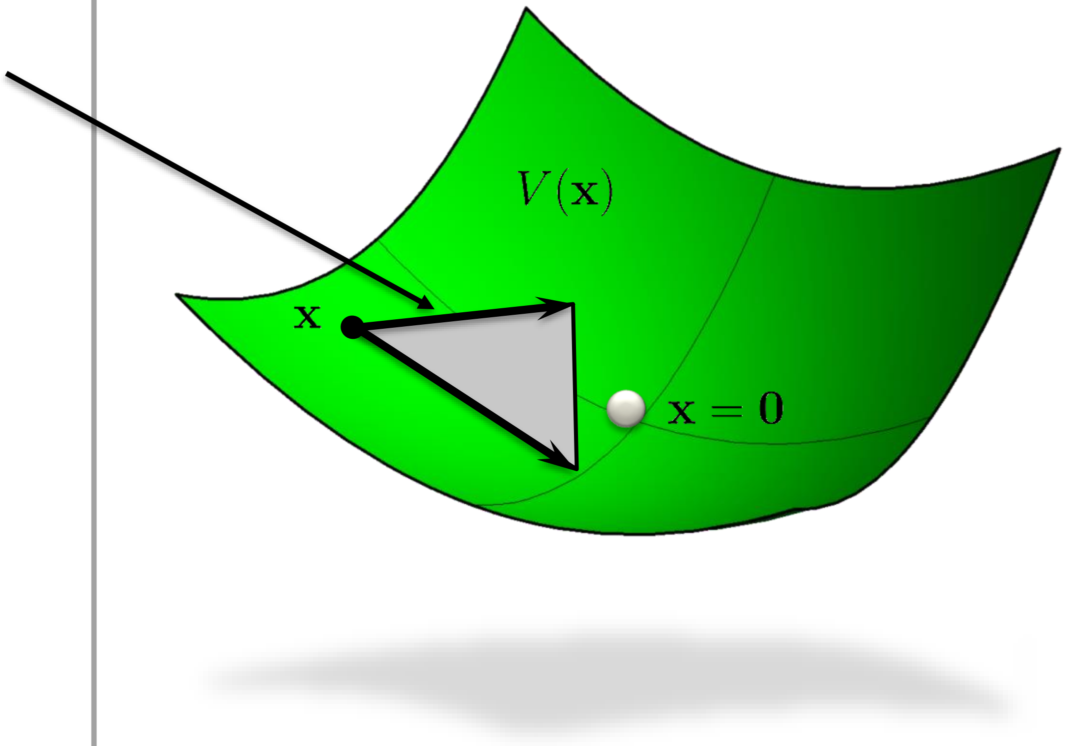
Caltech

Control Lyapunov Function

$$\alpha_1(\|\mathbf{x}\|_2) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|_2)$$

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \dot{V}(\mathbf{x}, \mathbf{u}) \leq -\alpha_3(\|\mathbf{x}\|_2)$$

$$\dot{V}(\mathbf{x}, \mathbf{u}) = \frac{\partial V}{\partial \mathbf{x}}(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u})$$



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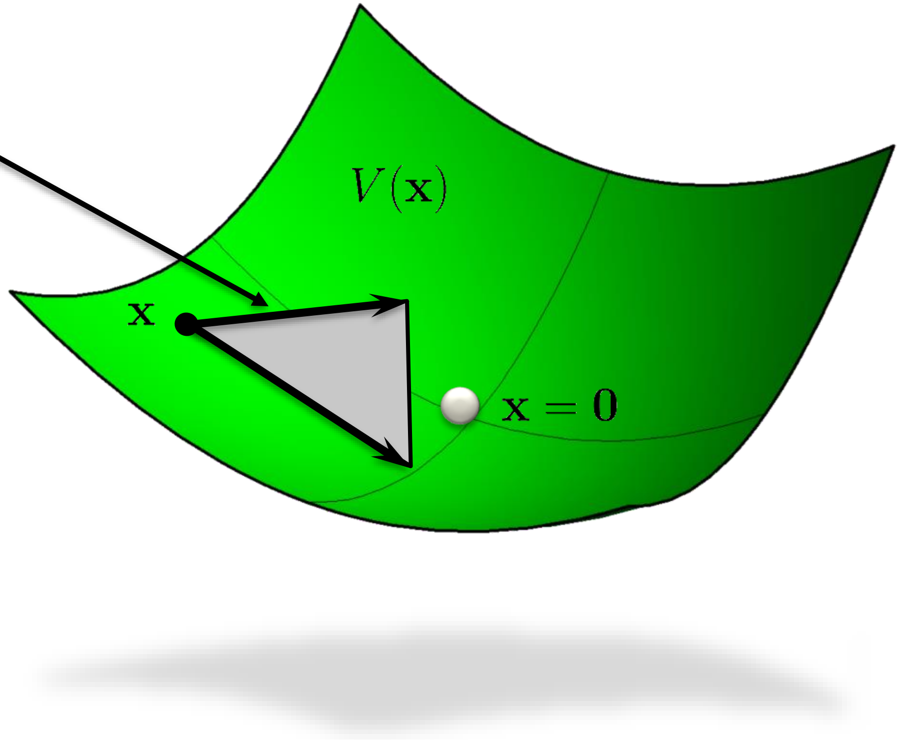
Feedback Controllers

[2] Z. Artstein, Stabilization with relaxed controls, 1983.
[3] E. Sontag, A universal construction of Artstein's theorem on nonlinear stabilization, 1989.
[4] R. Freeman, P. Kokotovic, Inverse Optimality in Robust Stabilization, 1996.



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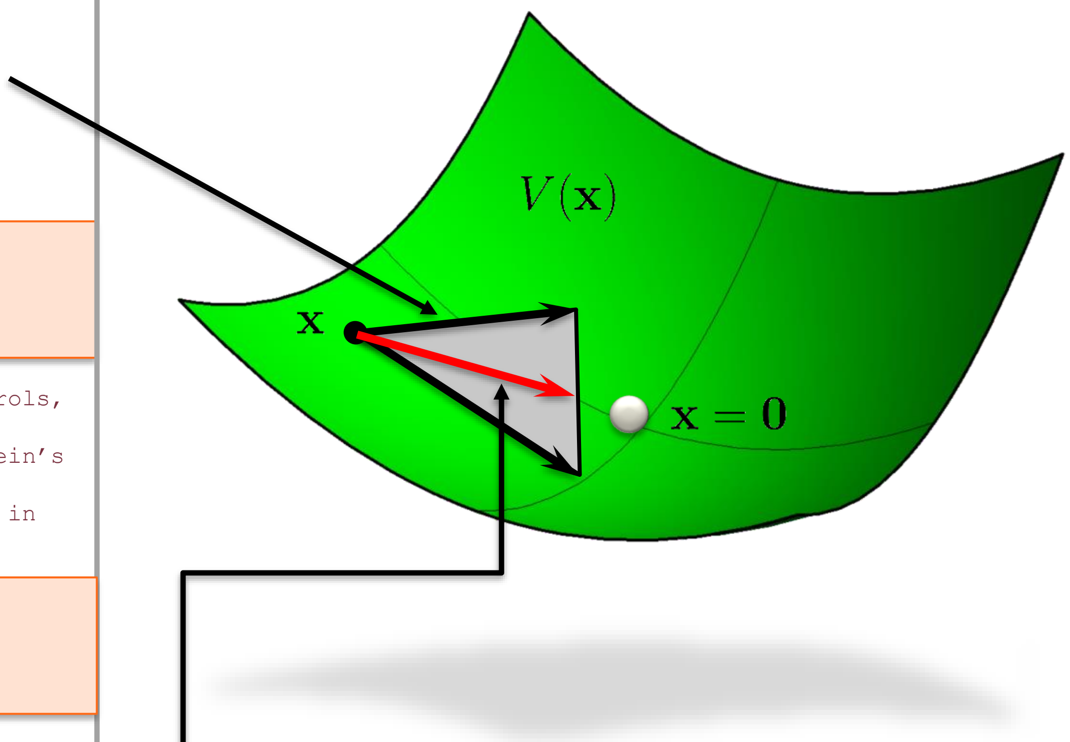
CLF Quadratic Program[5]

$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u}\in\mathbb{R}^m}{\operatorname{argmin}} \|\mathbf{u}\|_2$$

$$\text{s.t.} \quad \dot{V}(\mathbf{x},\mathbf{u}) \leq -\alpha_3(\|\mathbf{x}\|_2)$$



[5] A. Ames, M. Powell, Towards the unification of locomotion and manipulation through control lyapunov functions and quadratic programs.

Control Barrier Functions (CBFs)

Caltech

Control Barrier Function

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \mid h(\mathbf{x}) \geq 0\}$$

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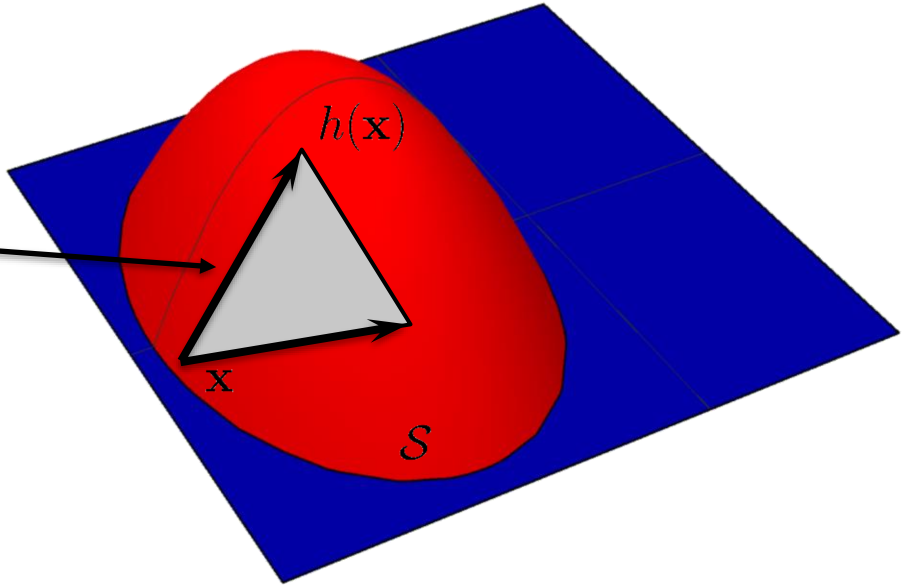

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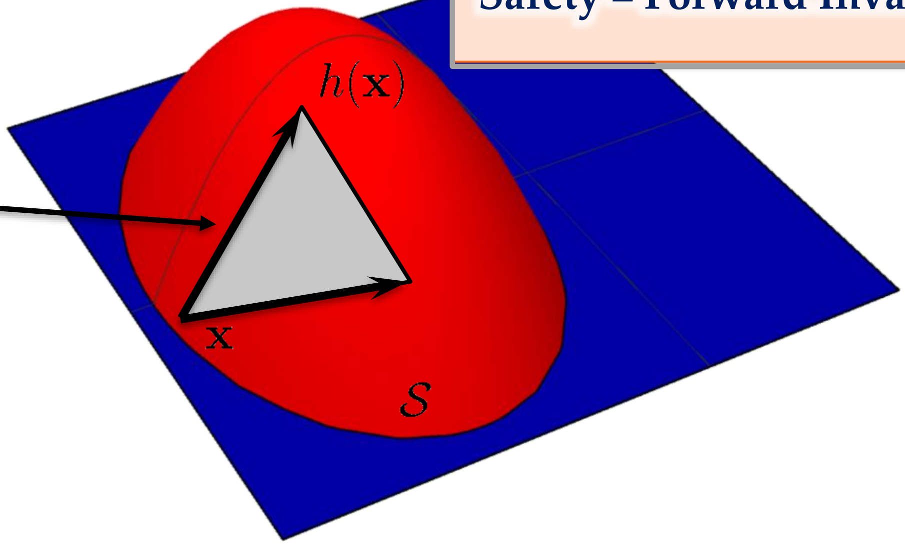
Safety = Forward Invariance



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[6] A. Ames, et al. Control barrier function based quadratic programs with application to adaptive cruise control, 2014.
[7] A. Ames, et al. Control barrier function based quadratic programs for safety critical systems, 2017.

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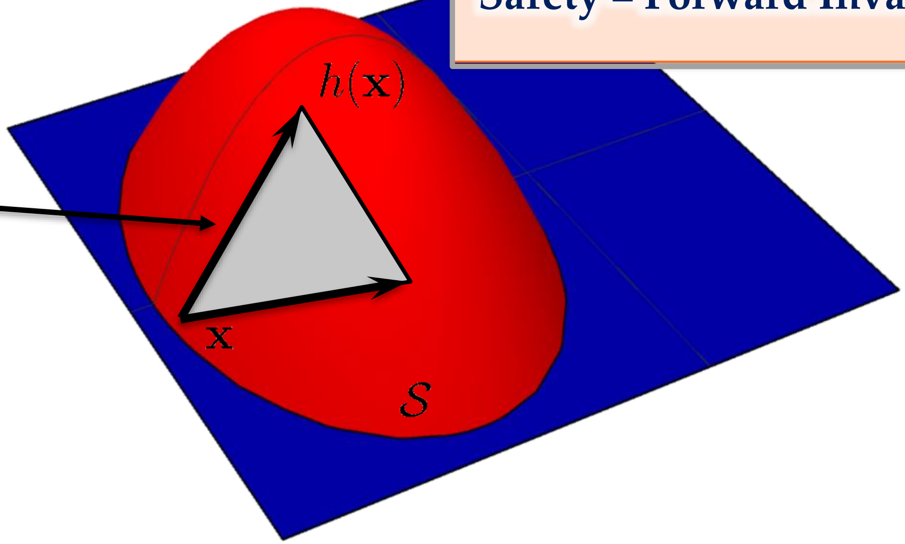

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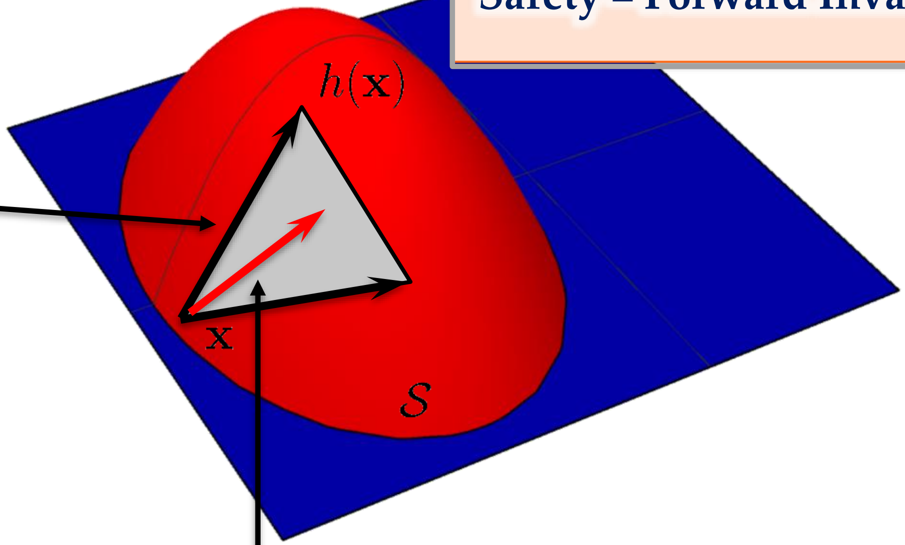

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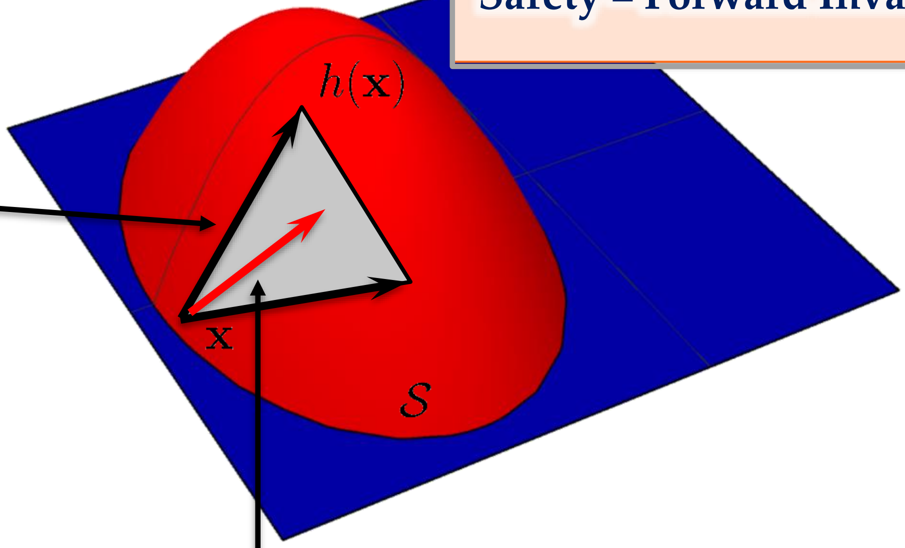

CLF-CBF Quadratic Program[8]

$$\mathbf{k}(\mathbf{x}) = \underset{(\mathbf{u}, \delta) \in \mathbb{R}^{m+1}}{\operatorname{argmin}} \|\mathbf{u}\|_2^2 + \delta^2$$
$$\text{s.t.} \quad \dot{h}(\mathbf{x}, \mathbf{u}) \geq -\alpha(h(\mathbf{x}))$$
$$\dot{V}(\mathbf{x}, \mathbf{u}) \leq -\alpha_3(\|\mathbf{x}\|) + \delta$$

[8] M. Jankovic, Robust control barrier functions for constrained stabilization of nonlinear systems, 2018.

Uncertain System Dynamics

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Equations of Motion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\boldsymbol{\theta}^* + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^{n \times p} \quad \boldsymbol{\theta}^* \in \mathbb{R}^p$$

True Dynamics

Physical System

Uncertain System Dynamics

Caltech

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Methods

- Adaptive Control[9]
- System Identification[10]
- Machine Learning[11]
- High-gain control[12]

True Dynamics

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[9] M. Krstic, et al., Nonlinear Adaptive Control Design
[10] L. Ljung, System Identification
[11] J. Kober, et al., Reinforcement learning in robotics: A survey
[12] A. Ilchmann, et al., High-gain control without identification: a survey

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Assumptions

$\mathbf{F}$ is locally Lipschitz continuous

$$\mathbf{F}(\mathbf{0}) = \mathbf{0}$$

True Dynamics

Physical System

Adaptive Control

Caltech

Adaptive Controller + Parameter Update Law	
$\mathbf{u} = \mathbf{k}(\mathbf{x}, \widehat{\boldsymbol{\theta}})$ $\dot{\widehat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\tau}(\mathbf{x}, \widehat{\boldsymbol{\theta}})$	$\widehat{\boldsymbol{\theta}} \in \mathbb{R}^p \quad \boldsymbol{\Gamma} \in \mathbb{R}^{p \times p}$ $\mathbf{k} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ $\boldsymbol{\tau} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$

Adaptive Control

Caltech

Adaptive Controller + Parameter Update Law

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$$\dot{\widehat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\tau}(\mathbf{x}, \widehat{\boldsymbol{\theta}})$$

$$\widehat{\boldsymbol{\theta}} \in \mathbb{R}^p \quad \boldsymbol{\Gamma} \in \mathbb{R}^{p \times p}$$

$$\mathbf{k} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$$

$$\boldsymbol{\tau} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$$

Assumptions

(A1) $\mathbf{k}$ locally Lipschitz continuous on $(\mathbb{R}^n \setminus \{\mathbf{0}\}) \times \mathbb{R}^p$

(A2) $\mathbf{k}(\mathbf{0}, \widehat{\boldsymbol{\theta}}) = \mathbf{0} \ \ \forall \widehat{\boldsymbol{\theta}} \in \mathbb{R}^p$

(A3) $\boldsymbol{\tau}$ locally Lipschitz continuous on $\mathbb{R}^n \times \mathbb{R}^p$

(A4) $\boldsymbol{\Gamma} \in \mathbb{R}^{p \times p}$ is symmetric and positive-definite

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(A4) $\boldsymbol{\Gamma} \in \mathbb{R}^{p\times p}$ is symmetric and positive-definite

Composite Dynamics

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\widehat{\boldsymbol{\theta}}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\boldsymbol{\theta}^* + \mathbf{g}(\mathbf{x})\mathbf{k}(\mathbf{x}, \widehat{\boldsymbol{\theta}}) \\ \boldsymbol{\Gamma}\tau(\mathbf{x}, \widehat{\boldsymbol{\theta}}) \end{bmatrix}$$

Adaptive Stabilization

Caltech

Global Adaptive Stabilizability

$$\mathbf{u} = \mathbf{k}(\mathbf{x}, \widehat{\boldsymbol{\theta}})$$

$$\dot{\widehat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\tau}(\mathbf{x}, \widehat{\boldsymbol{\theta}})$$

+(A1)-(A4)

Adaptive Stabilization

Caltech

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Adaptive Control Lyapunov Functions (aCLFs)[13]

$$\alpha_1(\|\mathbf{x}\|, \boldsymbol{\theta}) \leq V_a(\mathbf{x}, \boldsymbol{\theta}) \leq \alpha_2(\|\mathbf{x}\|, \boldsymbol{\theta})$$

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \frac{\partial V_a}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\theta}) \left(\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x}) \left(\boldsymbol{\theta} + \boldsymbol{\Gamma} \frac{\partial V_a}{\partial \boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}) \right) + \mathbf{g}(\mathbf{x})\mathbf{u} \right) \leq -\alpha_3(\|\mathbf{x}\|, \boldsymbol{\theta})$$

$$\alpha_i(\cdot, \boldsymbol{\theta}) \in \mathcal{K}_\infty \quad \forall \boldsymbol{\theta} \in \mathbb{R}^p$$

[13] M. Krstic, P.V. Kokotovic, Control Lyapunov functions for adaptive nonlinear stabilization, 1995.

Adaptive Stabilization

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$+$(A1)-(A4) $\Rightarrow$ $(\mathbf{x}(t), \widehat{\boldsymbol{\theta}}(t))$ globally bounded, $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$

⇓ ⇑

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Adaptive Stabilization

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Parameter Error

$$\widetilde{\theta} = \theta^* - \widehat{\theta}$$

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Parameter Error

$$\widetilde{\boldsymbol{\theta}} = \boldsymbol{\theta}^* - \widehat{\boldsymbol{\theta}}$$

Composite Lyapunov Function

$$V(\mathbf{x}, \widehat{\boldsymbol{\theta}}) = V_a(\mathbf{x}, \widehat{\boldsymbol{\theta}}) + \frac{1}{2}\widetilde{\boldsymbol{\theta}}^\top \boldsymbol{\Gamma}^{-1}\widetilde{\boldsymbol{\theta}}$$

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Update Law

$$\boldsymbol{\tau}(\mathbf{x}, \widehat{\boldsymbol{\theta}}) = \left(\frac{\partial V_a}{\partial \mathbf{x}}(\mathbf{x}, \widehat{\boldsymbol{\theta}})\mathbf{F}(\mathbf{x})\right)^\top$$

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LaSalle's Invariance Principle

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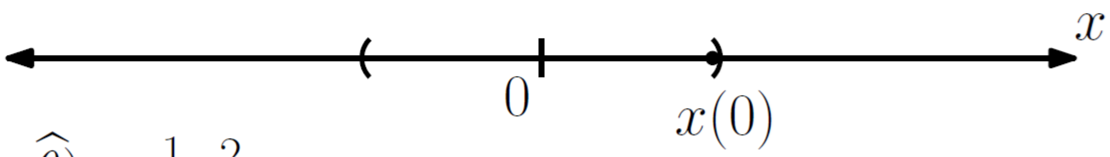
Only sublevel sets of V are forward invariant

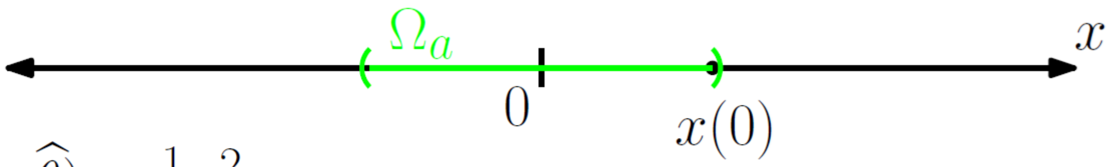
[13] M. Krstic, P.V. Kokotovic, Control Lyapunov functions for adaptive nonlinear stabilization, 1995.

Adaptive Stabilization



$$V_a(x, \widehat{\theta}) = \frac{1}{2}x^2$$

Adaptive Stabilization

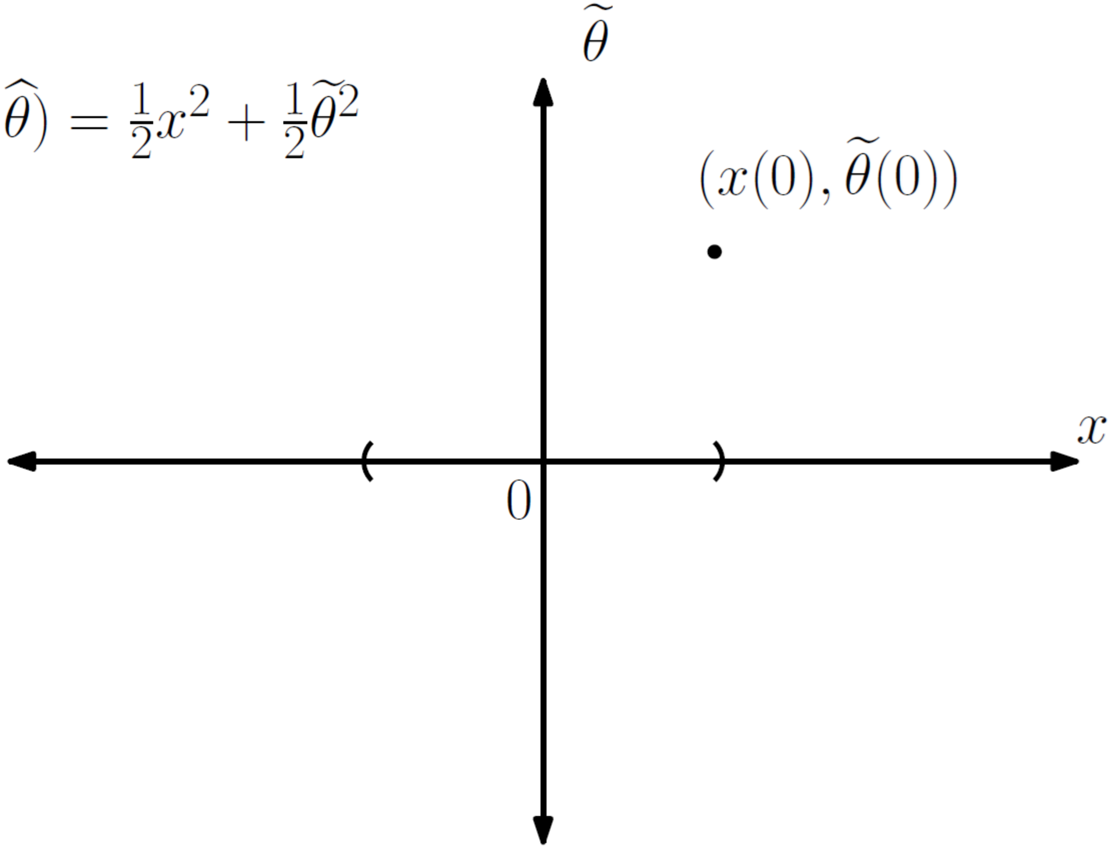


$$V_a(x, \widehat{\theta}) = \frac{1}{2}x^2$$

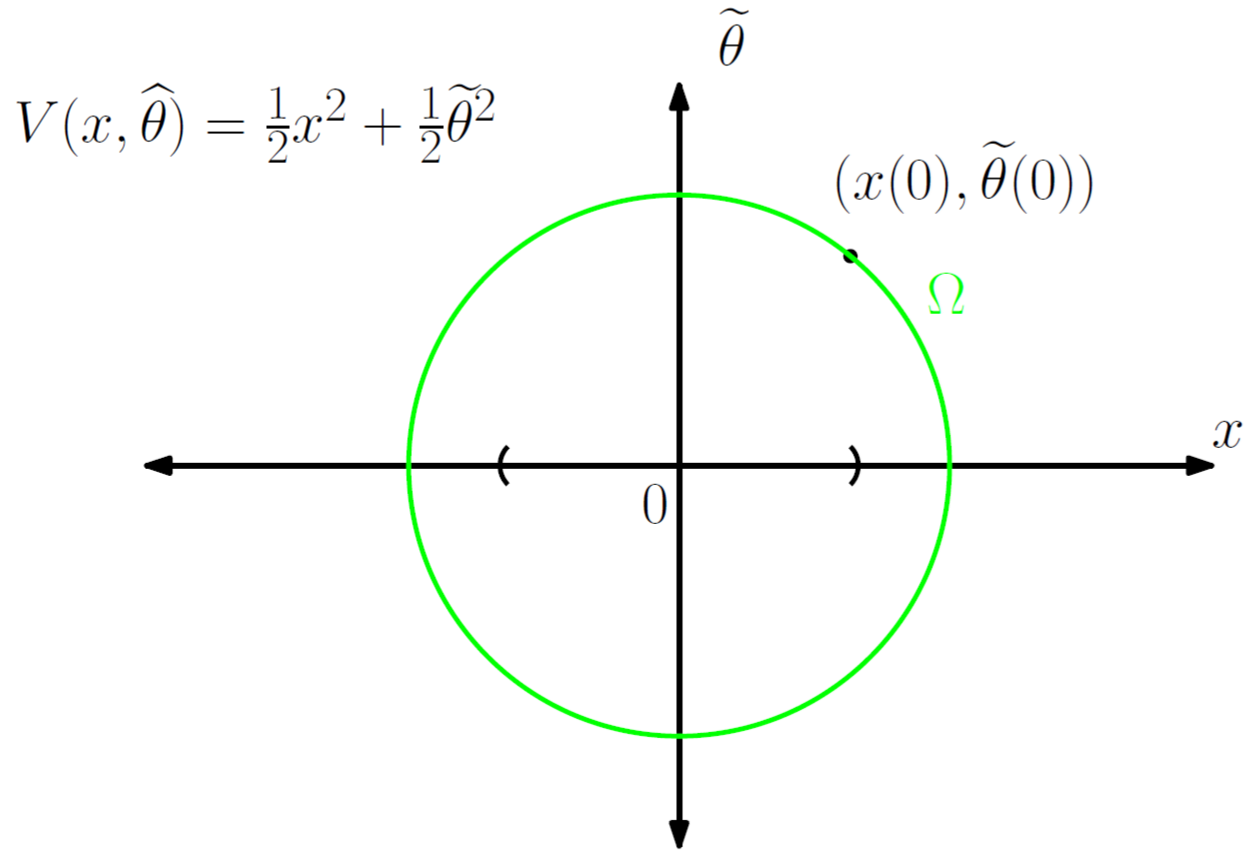
Adaptive Stabilization

Caltech

$$V(x, \widehat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}\widetilde{\theta}^2$$

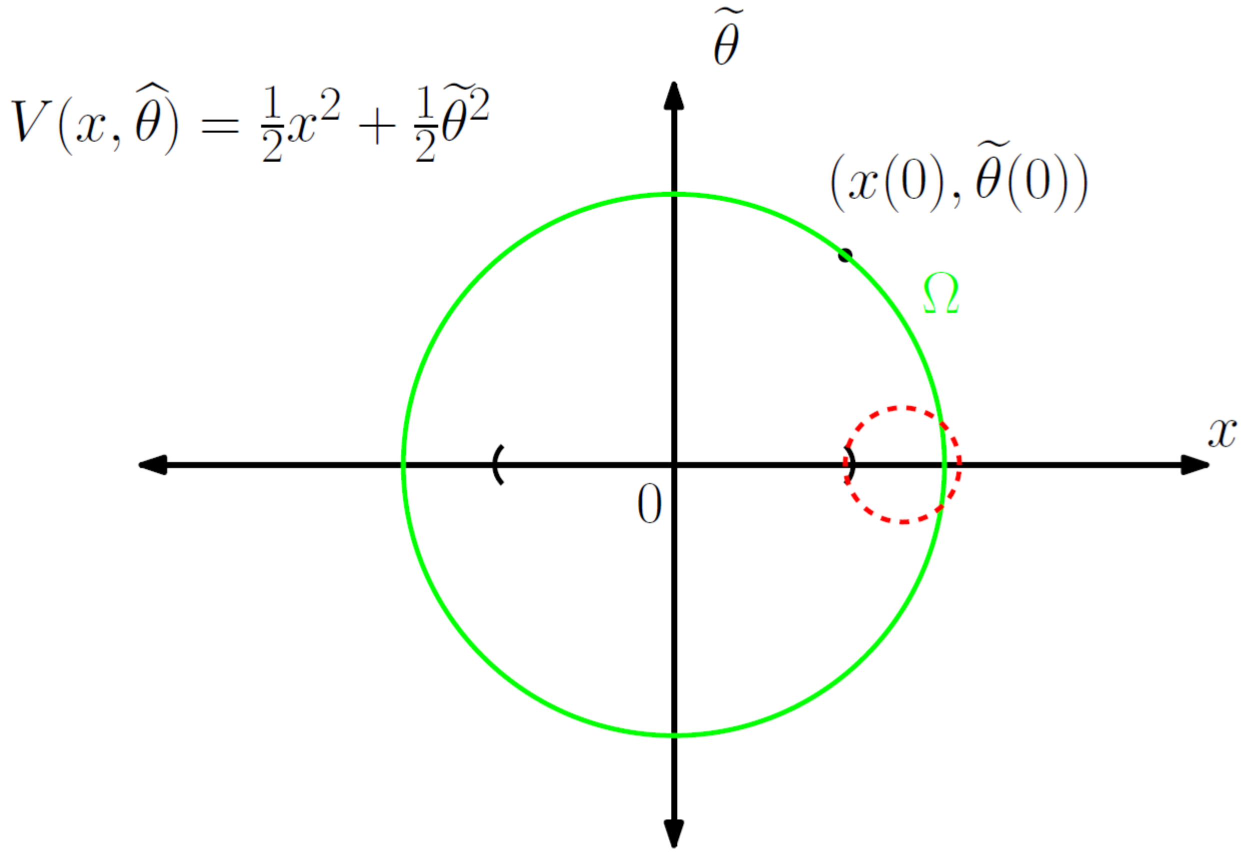


Adaptive Stabilization



Adaptive Stabilization

Caltech



Adaptive Safety

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Parametric Safe Sets

$$\mathcal{S}_{\boldsymbol{\theta}} \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid h_a(\mathbf{x}, \boldsymbol{\theta}) \geq 0\}$$

Adaptive Safety

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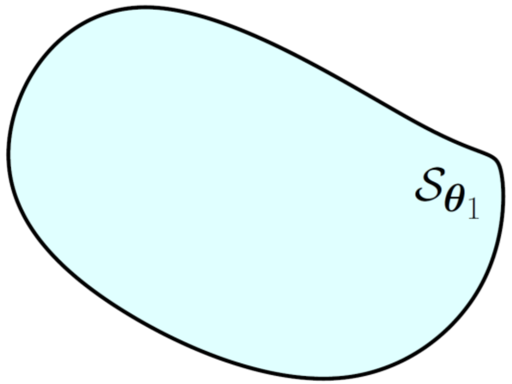

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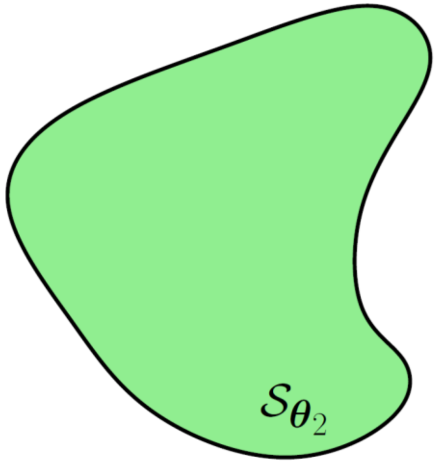

Adaptive Safety

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Adaptively Safe

$$\mathbf{u} = \mathbf{k}(\mathbf{x}, \widehat{\boldsymbol{\theta}})$$
$$\dot{\widehat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\tau}(\mathbf{x}, \widehat{\boldsymbol{\theta}})$$

+(A1)-(A4)

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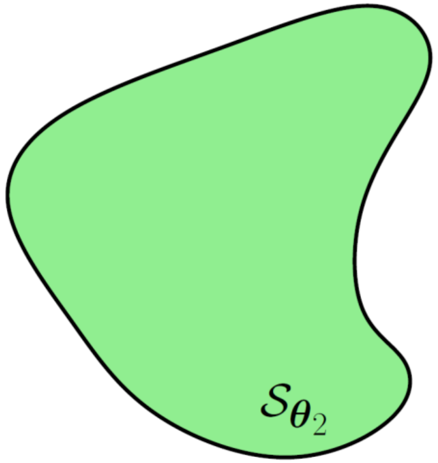

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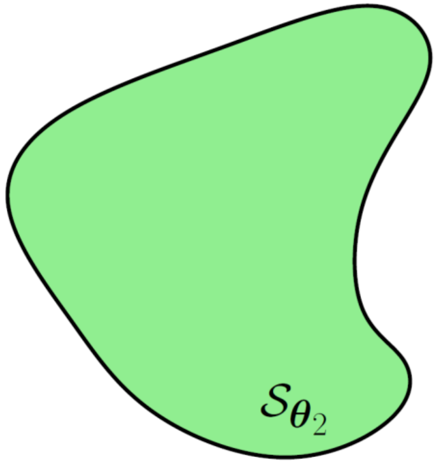

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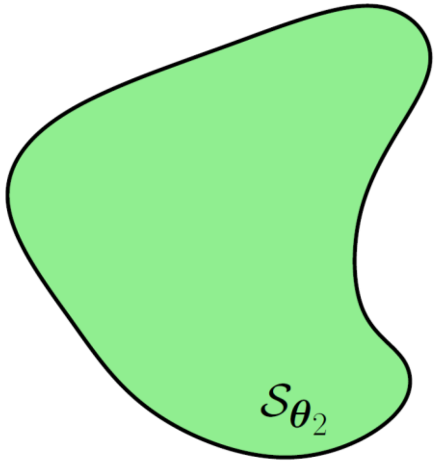

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Adaptive Control Barrier Functions (aCBFs)

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \frac{\partial h_a}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\theta}) \left(\mathbf{f}(\mathbf{x}) + \mathbf{F}(\mathbf{x}) \left(\boldsymbol{\theta} - \boldsymbol{\Gamma} \frac{\partial h_a}{\partial \boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}) \right) + \mathbf{g}(\mathbf{x})\mathbf{u} \right) \geq 0$$

Adaptive Safety

Caltech

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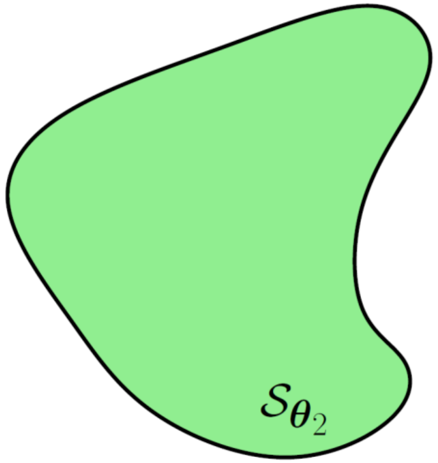

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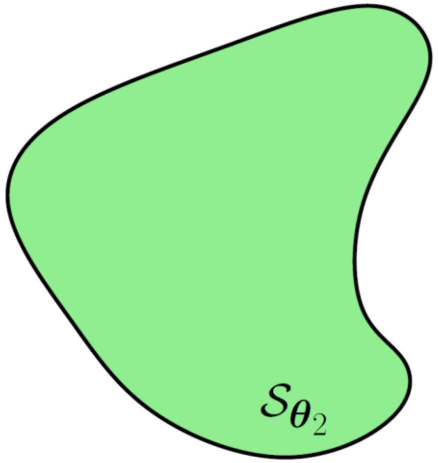

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Adaptive Safety

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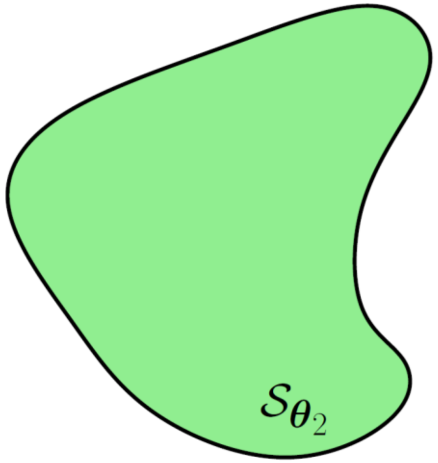

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Adaptive Safety

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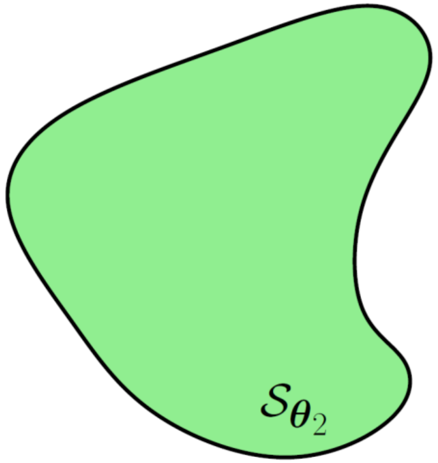

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Adaptive Safety

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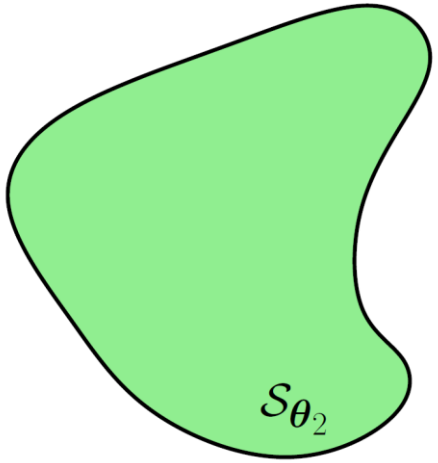

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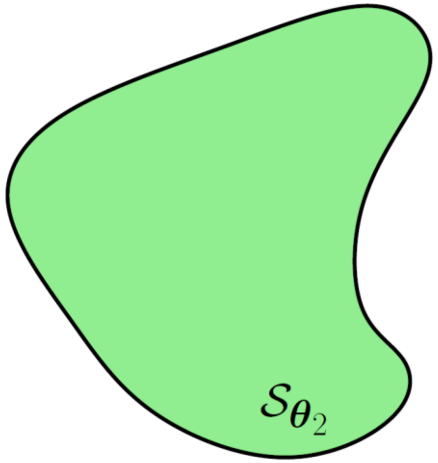

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[14] S. Prajna, Optimization-based methods for nonlinear and hybrid systems verification, 2005.
[15] P. Wieland and F. Allgöwer, Constructive safety using control barrier functions, 2007.

Adaptive Safety

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Composite Barrier Function

$$h(\mathbf{x}, \widehat{\boldsymbol{\theta}}) = h_a(\mathbf{x}, \widehat{\boldsymbol{\theta}}) - \frac{1}{2}\widetilde{\boldsymbol{\theta}}^\top \boldsymbol{\Gamma}^{-1}\widetilde{\boldsymbol{\theta}}$$

Adaptive Safety

Caltech

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↑

Assumptions

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Adaptive Safety

Caltech

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Initial Safety

$$h(\mathbf{x}_0, \widehat{\boldsymbol{\theta}}_0) \geq 0$$

Assumptions

$$\mathbf{x}_0 \in \text{int}(\mathcal{S}_{\widehat{\boldsymbol{\theta}}_0})$$

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Update Law

$$\boldsymbol{\tau}(\mathbf{x}, \widehat{\boldsymbol{\theta}}) = -\left(\frac{\partial h_a}{\partial \mathbf{x}}(\mathbf{x}, \widehat{\boldsymbol{\theta}})\mathbf{F}(\mathbf{x})\right)^\top$$

Assumptions

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Composite Invariance

$$h(\mathbf{x}(t),\widehat{\boldsymbol{\theta}}(t)) \geq 0$$

Assumptions

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Adaptive Safety

$$h_a(\mathbf{x}(t),\widehat{\boldsymbol{\theta}}(t)) \geq 0$$

Understanding aCBF Conservativeness

Caltech

Why not $\dot{h}_a(\mathbf{x}, \widehat{\boldsymbol{\theta}}, \mathbf{u}) \geq -\alpha(h_a(\mathbf{x}, \widehat{\boldsymbol{\theta}}))$ **?**

[16] M. Nagumo, Über die lage der integralkurven gewöhnlicher differentialgleichungen, 1942.

Understanding aCBF Conservativeness

Caltech

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Pathological Example

$$\dot{x} = \theta + u$$

$$h_a(x) = 1 - x^2$$

$$h(x, \hat{\theta}) = h_a(x) - \tfrac{1}{2}\gamma^{-1}\tilde{\theta}^2$$

Understanding aCBF Conservativeness

Caltech

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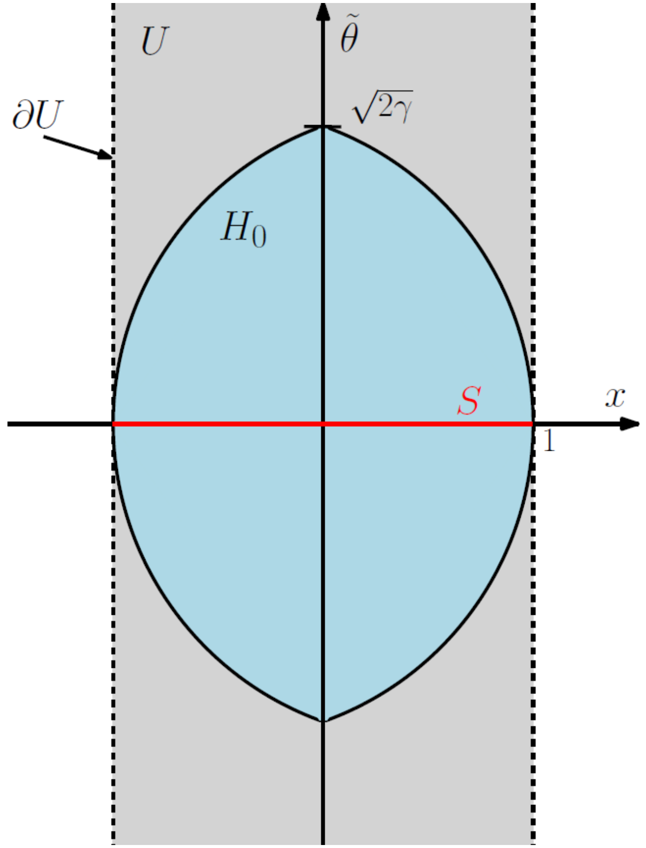
$$\mathcal{S} \triangleq \{x \in \mathbb{R} \mid x^2 \leq 1\}$$
$$U \triangleq \{(x, \tilde{\theta}) \in \mathbb{R}^2 \mid x \in \mathcal{S}\}$$
$$H_0 \triangleq \{(x, \tilde{\theta}) \in \mathbb{R}^2 \mid h(x, \hat{\theta}) \geq 0\}$$



Understanding aCBF Conservativeness

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Liénard System

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} \tilde{\theta} + \frac{1}{2} x \alpha(h_a(x)) \\ -2\gamma x \end{bmatrix}$$

Understanding aCBF Conservativeness

Caltech

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Liénard's Theorem[17]

Theorem 4 (Liénard's Theorem)**.** *Under the assumption that* $F, g \in C^1(\mathbb{R})$*,* F *and* g *are odd functions of* x*,* $xg(x) > 0$ *for* $x \neq 0$*,* $F(0) = 0$*,* $F'(0) < 0$*,* F *has single positive zero at* $x = a$*, and* F *increases monotonically to infinity for* $x \geq a$ *as* $x \to \infty$*, it follows that the Liénard system has exactly one limit cycle* $\mathbf{\Phi}$ *and it is stable.*

Corollary 1. *The stable limit cycle* Φ *is symmetric about the origin and passes through a point, denoted as* $P_2 = (x_2, \tilde{\theta}_2)$*, such that* $x_2 > a$*.*

[17] L. Perko, Differential equations and dynamical systems, 2013.

Understanding aCBF Conservativeness

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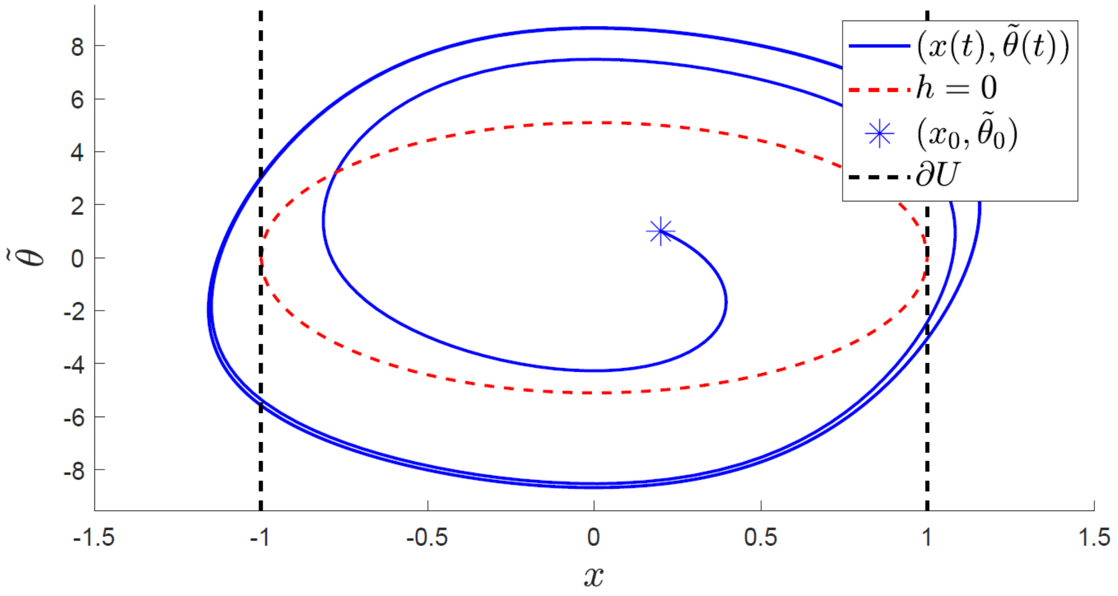

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Adaptive Cruise Control

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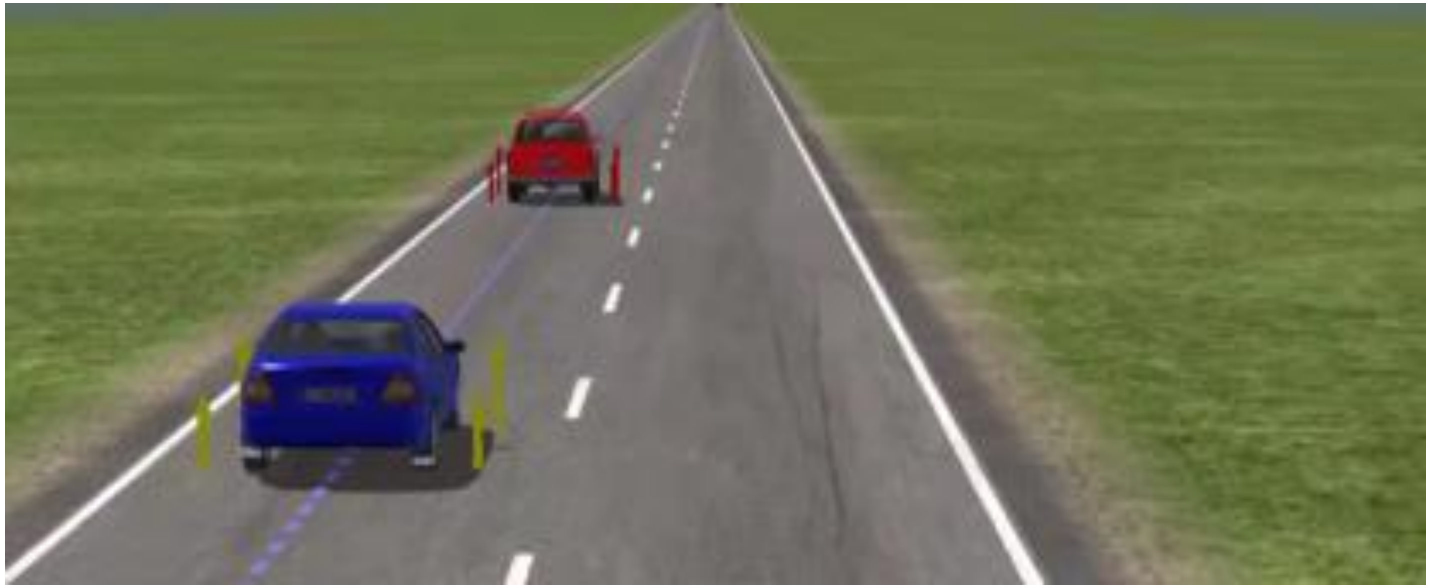

Adaptive Cruise Control

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States and Parameters

Vehicle velocity: v
Vehicle distance gap: D
Lead vehicle velocity: v_0
Vehicle mass: m
Unknown friction coefficients: f_0, f_1, f_2

Dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ v_0 - v \end{bmatrix} - \frac{1}{m}\begin{bmatrix} 1 & v & v^2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} u$$

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Safety Objective

$$D \geq 1.8v$$

Adaptive Cruise Control

Caltech

States and Parameters

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Unknown friction coefficients: f_0, f_1, f_2

Dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ v_0 - v \end{bmatrix} - \frac{1}{m}\begin{bmatrix} 1 & v & v^2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} u$$

Safety Objective

$$D \geq 1.8v$$

Adaptive CBF

$$h_a(v, D) = \begin{cases} \alpha^2 & \text{if } D - 1.8v \geq \alpha \\ \alpha^2 - (D - 1.8v - \alpha)^2 & \text{if } D - 1.8v < \alpha \end{cases}$$

Adaptive Cruise Control

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States and Parameters

Vehicle velocity: v
Vehicle distance gap: D
Lead vehicle velocity: v_0
Vehicle mass: m
Unknown friction coefficients: f_0, f_1, f_2

Dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ v_0 - v \end{bmatrix} - \frac{1}{m}\begin{bmatrix} 1 & v & v^2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} u$$

Safety Objective

$$D \geq 1.8v$$

Adaptive CBF

$$h_a(v, D) = \begin{cases} \alpha^2 & \text{if } D - 1.8v \geq \alpha \\ \alpha^2 - (D - 1.8v - \alpha)^2 & \text{if } D - 1.8v < \alpha \end{cases}$$

aCLF-aCBF-QP

$$\begin{aligned} \mathbf{k}(\mathbf{x}, \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\psi}}) = \operatorname{argmin} \quad & J(u) + c_V \delta_V^2 + c_p \delta_p^2 \\ \text{s.t.} \quad & \dot{V}_a(\mathbf{x}, \widehat{\boldsymbol{\theta}}, \mathbf{u}) \leq -\alpha_3(\|\mathbf{x}\|, \widehat{\boldsymbol{\theta}}) + \delta_V \\ & \dot{h}_a(\mathbf{x}, \widehat{\boldsymbol{\psi}}, \mathbf{u}) \geq 0 \\ & u \leq u_{max} + \delta_p \\ & u \geq -u_{max} - \delta_p \end{aligned}$$

$$\dot{\widehat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}_1 \left(\frac{\partial V_a}{\partial \mathbf{x}}(\mathbf{x}, \widehat{\boldsymbol{\theta}})\mathbf{F}(\mathbf{x})\right)^\top$$

$$\dot{\widehat{\boldsymbol{\psi}}} = -\boldsymbol{\Gamma}_2 \left(\frac{\partial h_a}{\partial \mathbf{x}}(\mathbf{x}, \widehat{\boldsymbol{\theta}})\mathbf{F}(\mathbf{x})\right)^\top$$

Adaptive Cruise Control

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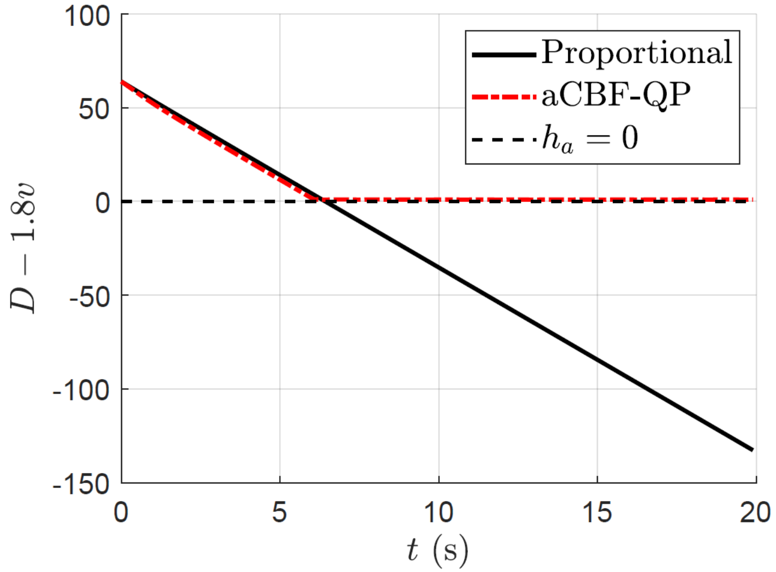

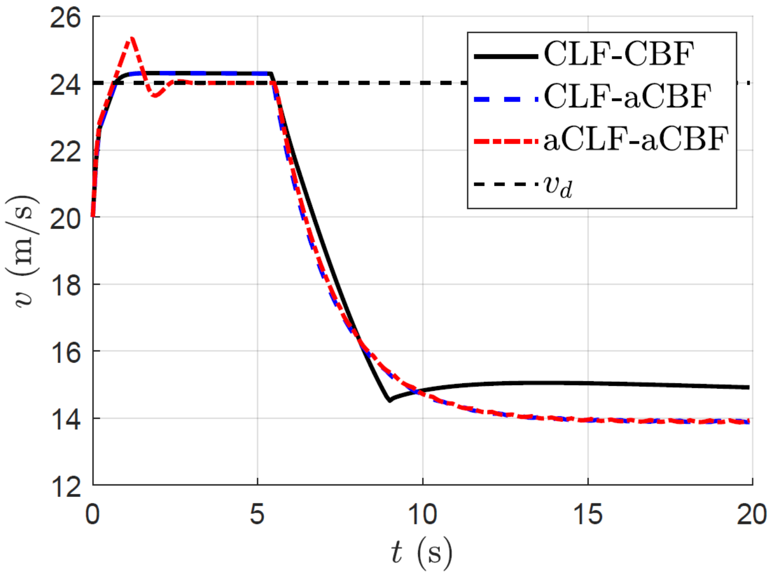

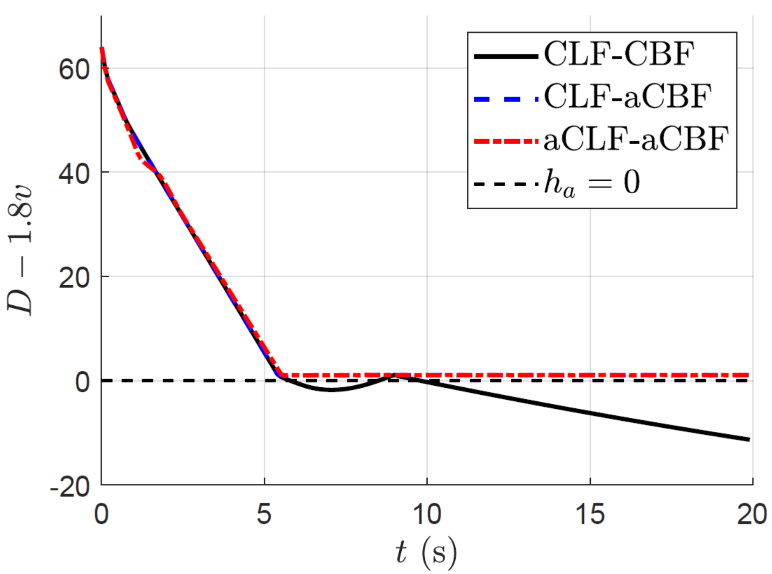

Conclusions

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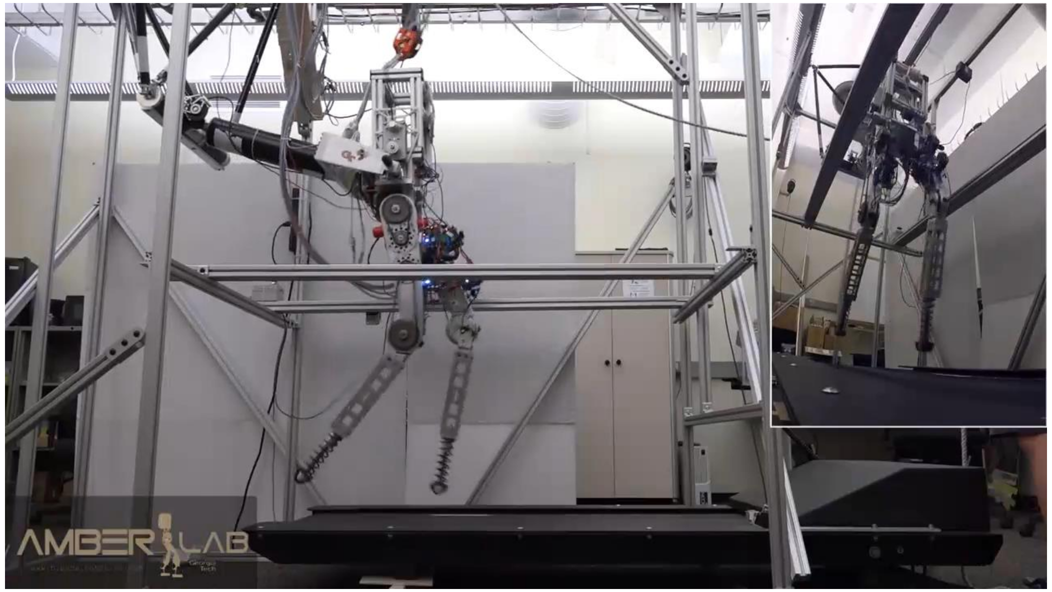
- **Adaptive Control Barrier Functions** offer solution for safety in the presence of parametric uncertainty

- Adaptive set invariance faces challenges not encountered by traditional adaptive stabilization methods

- Stabilization and safety with adaptation can be achieved simultaneously using aCLFs and aCBFs



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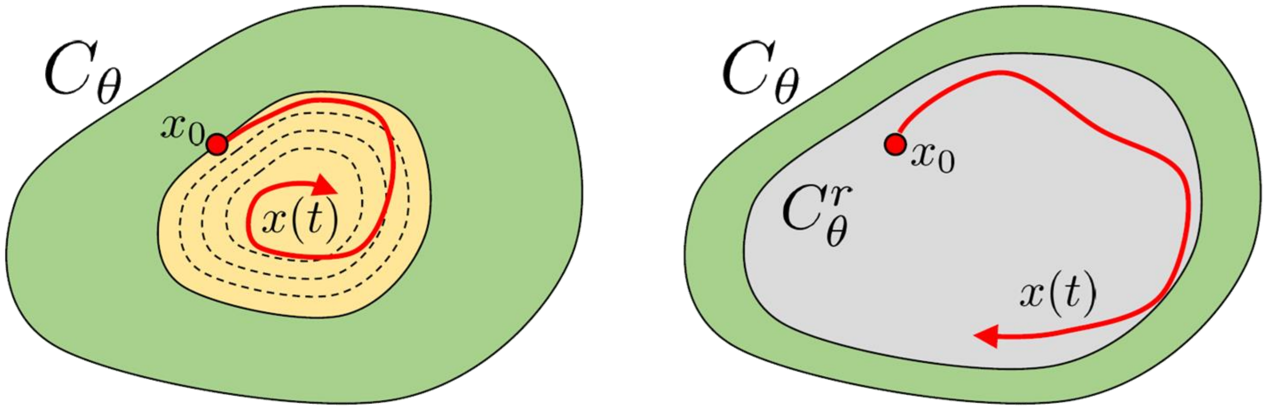
Robust Adaptive Control Barrier Functions: An Adaptive & Data-Driven Approach to Safety (Extended Version)
Brett T. Lopez, Jean-Jacques E. Slotine, Jonathan P. How
arXiv preprint arXiv:2003.10028, 2020.



(a) Safe set with adaptive control barrier functions (aCBFs).

(b) Safe set with robust adaptive control barrier functions (RaCBFs).

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Thank You!

Adaptive Safety with Control Barrier Functions

Andrew Taylor Aaron Ames