A Control Lyapunov Perspective on Episodic Learning via Projection to State Stability

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December 11th, 2019

Control in the real world is hard

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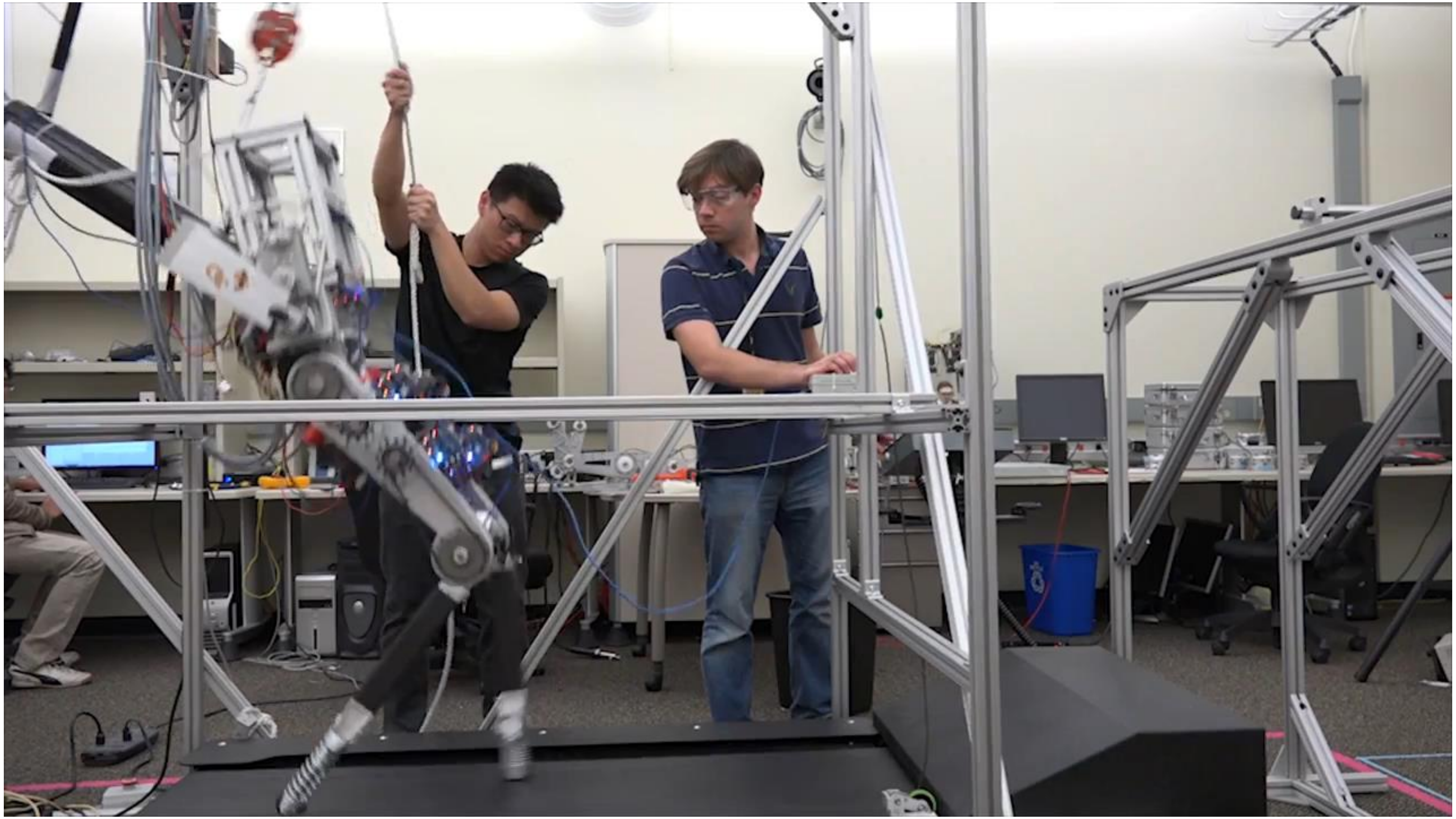
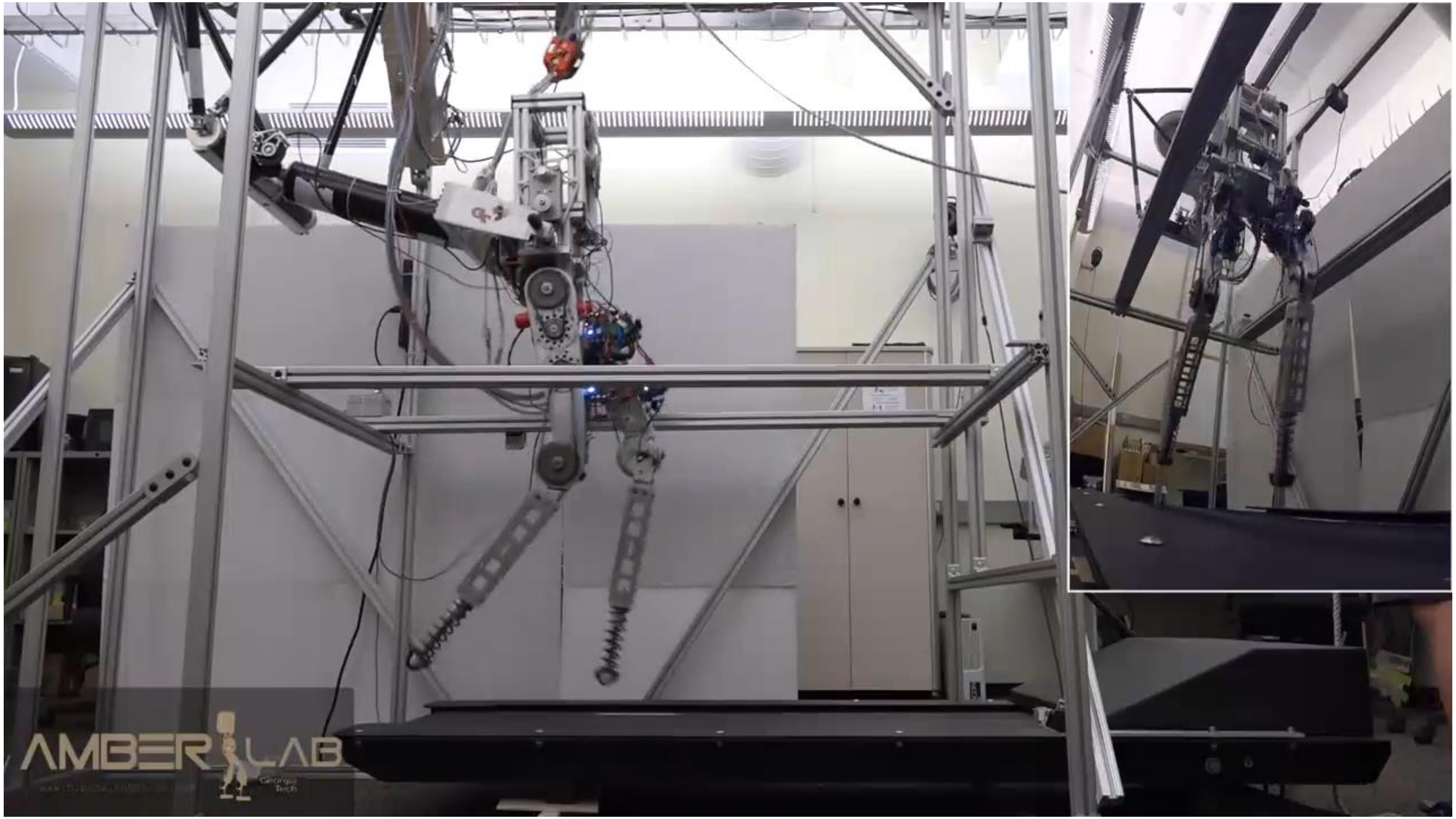


But: Pretty when it works…



W. Ma, et al., Bipedal robotic running with durus-2d: Bridging the gap between theory and experiment

Claim: Need to Bridge the Gap

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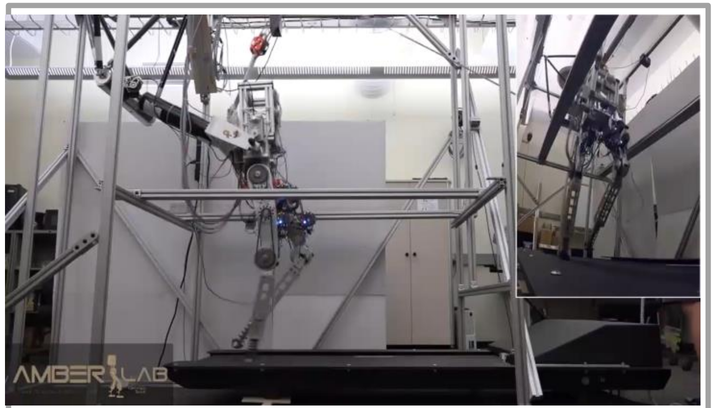
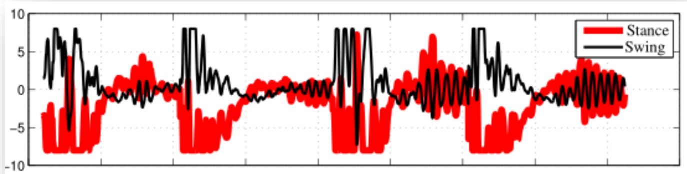
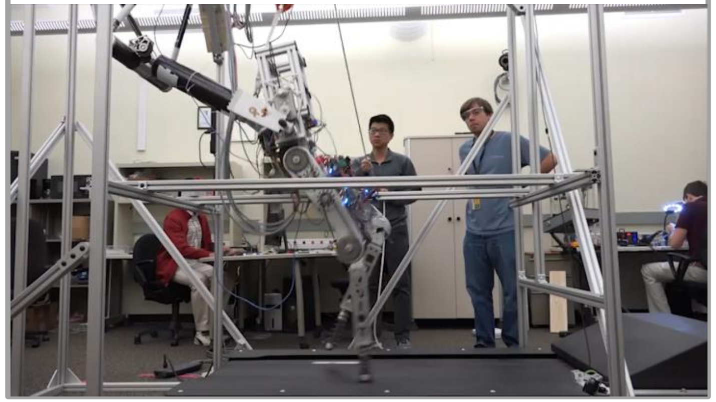


$$\mathbf{k}(\mathbf{x}) = \underset{\mathbf{u} \in \mathbb{R}^m}{\operatorname{argmin}} \; \|\mathbf{u}\|_2$$
$$\text{s.t.} \quad \dot{V}(\mathbf{x}, \mathbf{u}) \leq -\alpha_3\left(\|\mathbf{x}\|_2\right)$$

Bridge the Gap





Theorems & Proofs

Experimental Realization

Contributions

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- Framework for studying impact of disturbances in a projected environment via **Projection-to-State Stability (PSS)**
- Apply PSS to study how error in machine learning models estimating dynamics leads to degradation in stability guarantees
- Data driven method for bounding residual error in machine learning models after learning for affine control systems

System Dynamics

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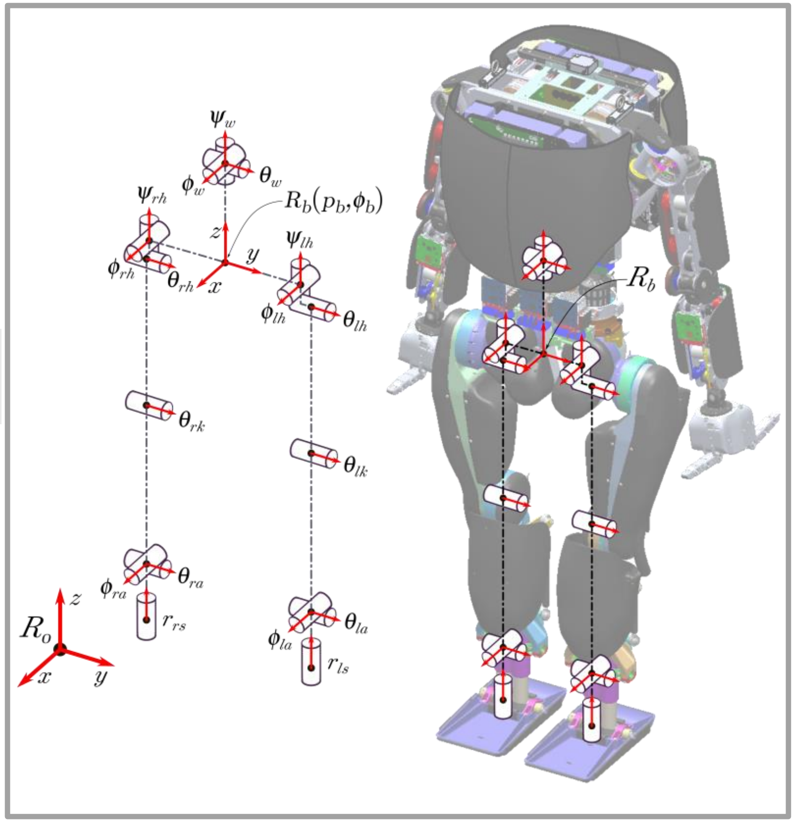
Equations of Motion

$$\hat{\dot{\mathbf{x}}} = \hat{\mathbf{f}}(\mathbf{x}) + \hat{\mathbf{g}}(\mathbf{x})\mathbf{u}$$

$$\mathbf{x} \in \mathbb{R}^n \qquad \mathbf{u} \in \mathbb{R}^m$$

$$\hat{\mathbf{f}} : \mathbb{R}^n \to \mathbb{R}^n \qquad \hat{\mathbf{g}} : \mathbb{R}^n \to \mathbb{R}^{n \times m}$$

Mathematical Model



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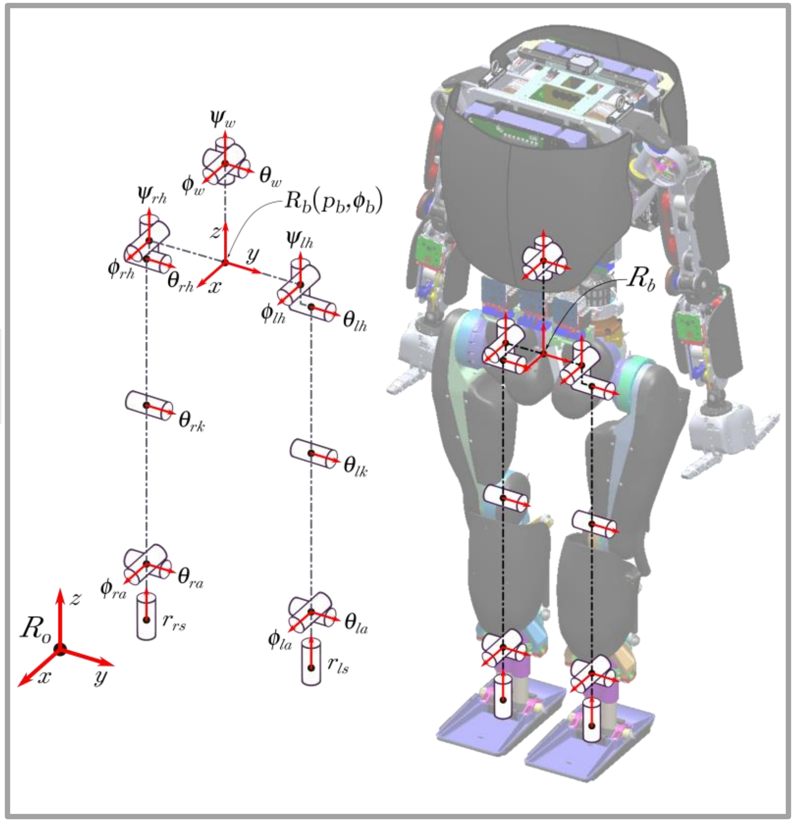
Assumptions

$\widehat{\mathbf{f}}, \widehat{\mathbf{g}}$ locally Lipschitz continuous

$$\exists\, \widehat{\mathbf{u}}_0 \in \mathbb{R}^m \text{ s.t. } \widehat{\mathbf{f}}(\mathbf{0}) + \widehat{\mathbf{g}}(\mathbf{0})\widehat{\mathbf{u}}_0 = \mathbf{0}$$

Mathematical Model



System Model

Control Lyapunov Functions (CLFs)

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Control Lyapunov Function

$$\alpha_1(\|\mathbf{x}\|_2) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|_2)$$

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \dot{V}(\mathbf{x}, \mathbf{u}) \leq -\alpha_3(\|\mathbf{x}\|_2)$$

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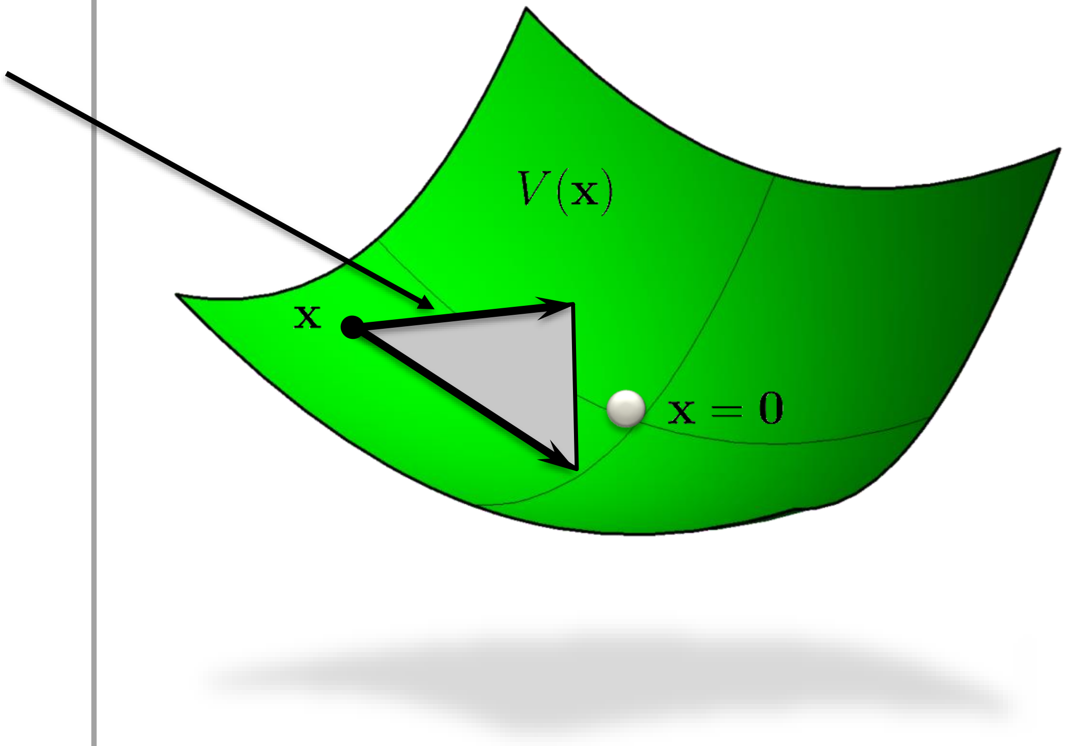

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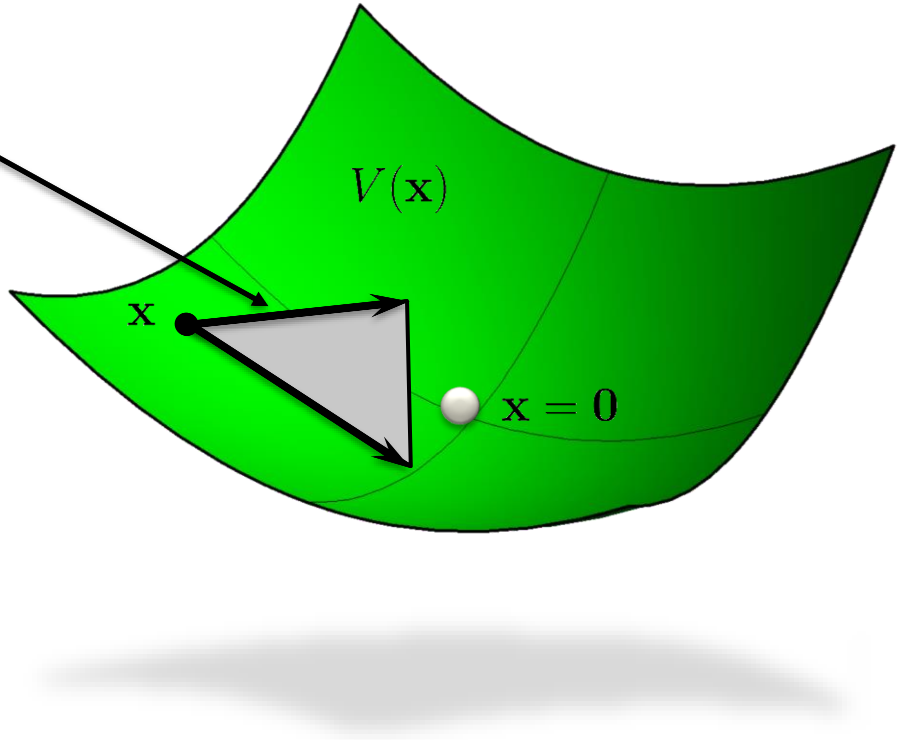
Feedback Controllers

[1] Z. Artstein, Stabilization with relaxed controls, 1983.
[2] E. Sontag, A universal construction of Artstein's theorem on nonlinear stabilization, 1989.
[3] R. Freeman, P. Kokotovic, Inverse Optimality in Robust Stabilization, 1996.



Control Lyapunov Functions (CLFs)

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CLF Quadratic Program[4]

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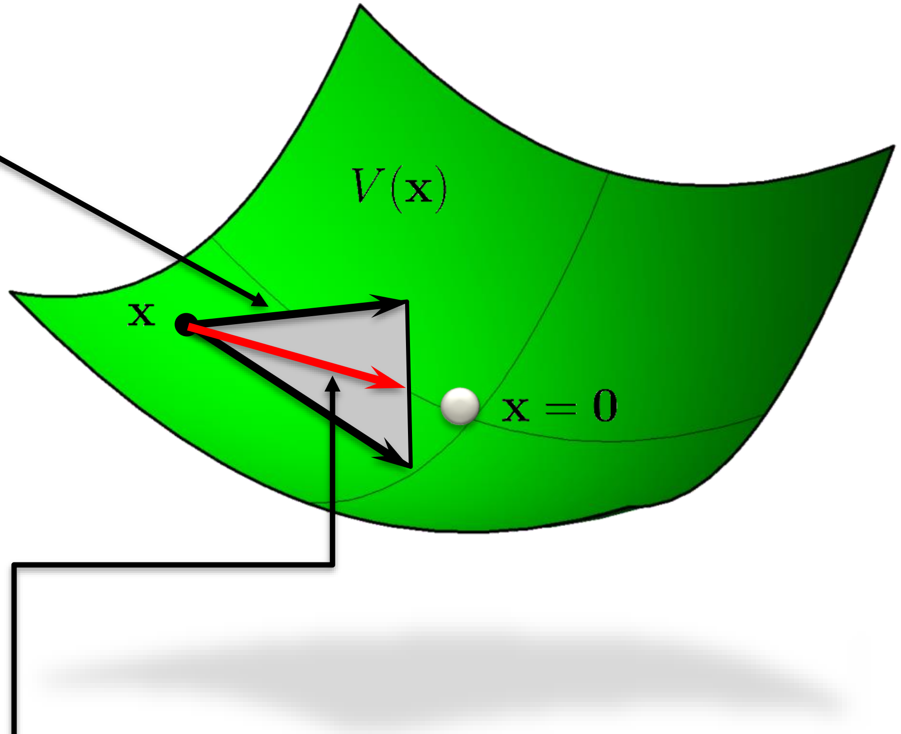

[4] A. Ames, M. Powell, Towards the unification of locomotion and manipulation through control lyapunov functions and quadratic programs.

Uncertain System Dynamics

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Equations of Motion

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True Dynamics

Physical Robot

Uncertain System Dynamics

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Methods

- Adaptive Control [1]
- System Identification [2]
- Machine Learning [3]
- High-gain control [4]

True Dynamics

Physical Robot

[1] M. Krstic, et al., Nonlinear Adaptive Control Design
[2] L. Ljung, System Identification
[3] J. Kober, et al., Reinforcement learning in robotics: A survey
[4] A. Ilchmann, et al., High-gain control without identification: a survey

Uncertain System Dynamics

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True Dynamics

Physical Robot

[1] A. Taylor, Episodic Learning with CLFs for Uncertain Robotic Systems, 2019

CLF Derivative Uncertainty

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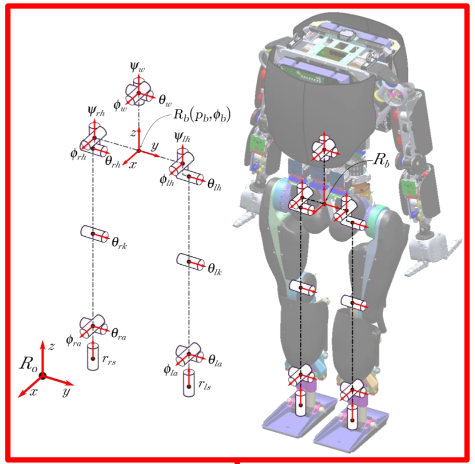
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

CLF Derivative Uncertainty



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$$\pm\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right)$$

CLF Derivative Uncertainty

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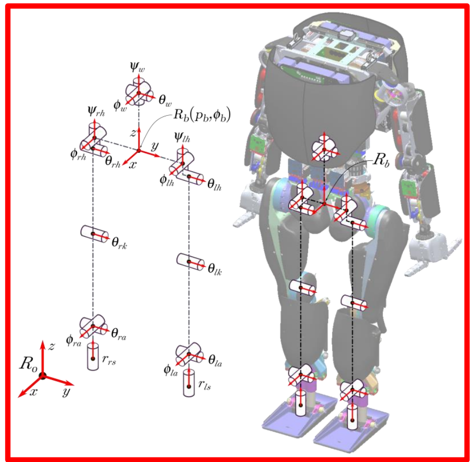

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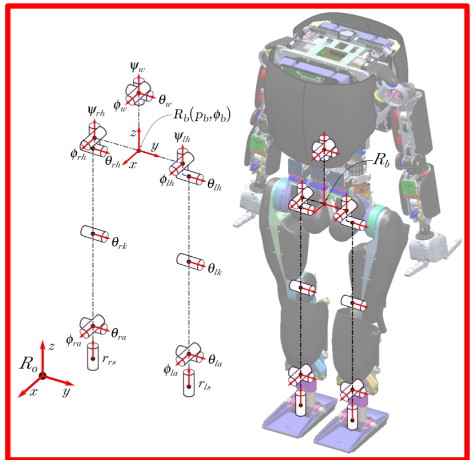

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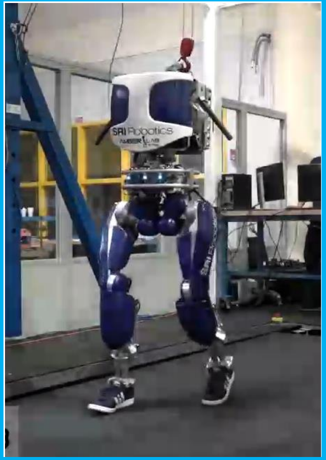

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CLF Derivative Uncertainty

Learn the dynamics

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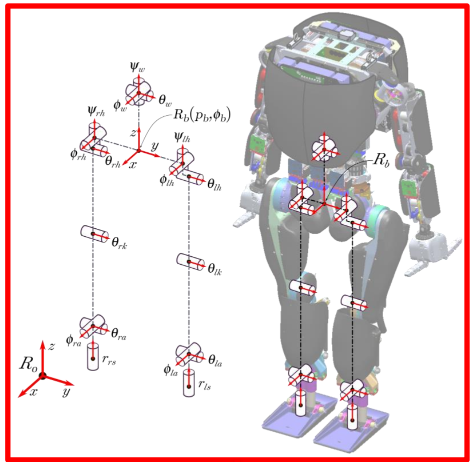

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CLF Derivative Uncertainty

Learn the residual dynamics

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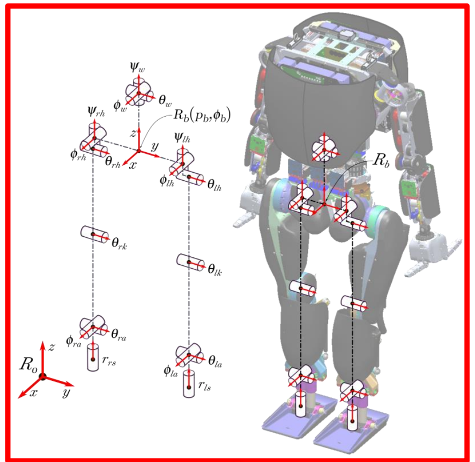

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

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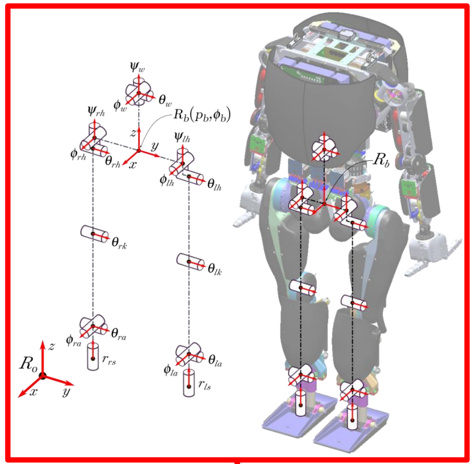
CLF Derivative Uncertainty

Learn the residual CLF dynamics



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Learning Control Lyapunov Functions

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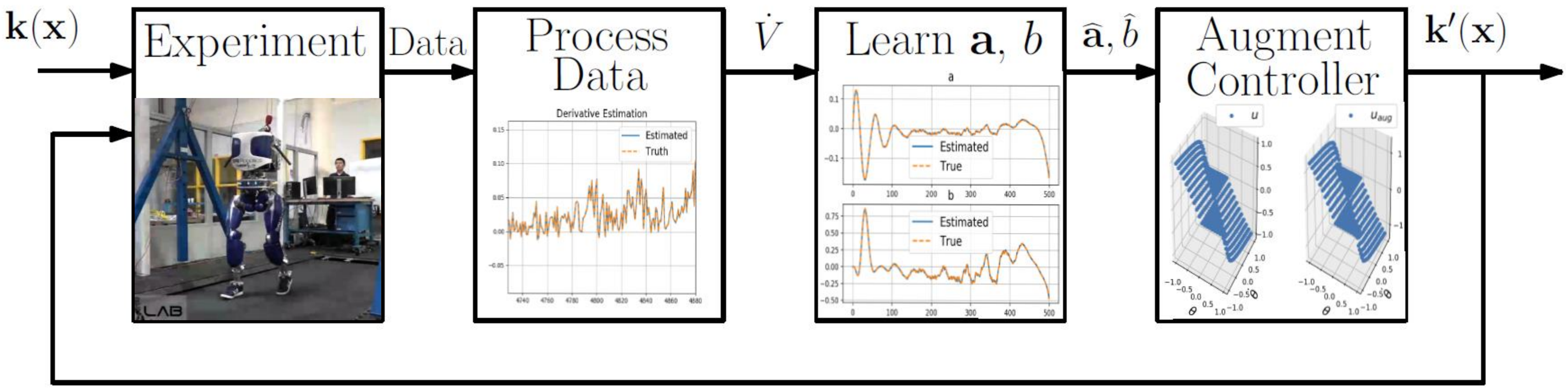
CLF Derivative Estimator

$$\dot{V}(\mathbf{x},\mathbf{u}) \approx \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x})$$



A. Taylor, Episodic Learning with CLFs
for Uncertain Robotic Systems, 2019

Learning Control Lyapunov Functions

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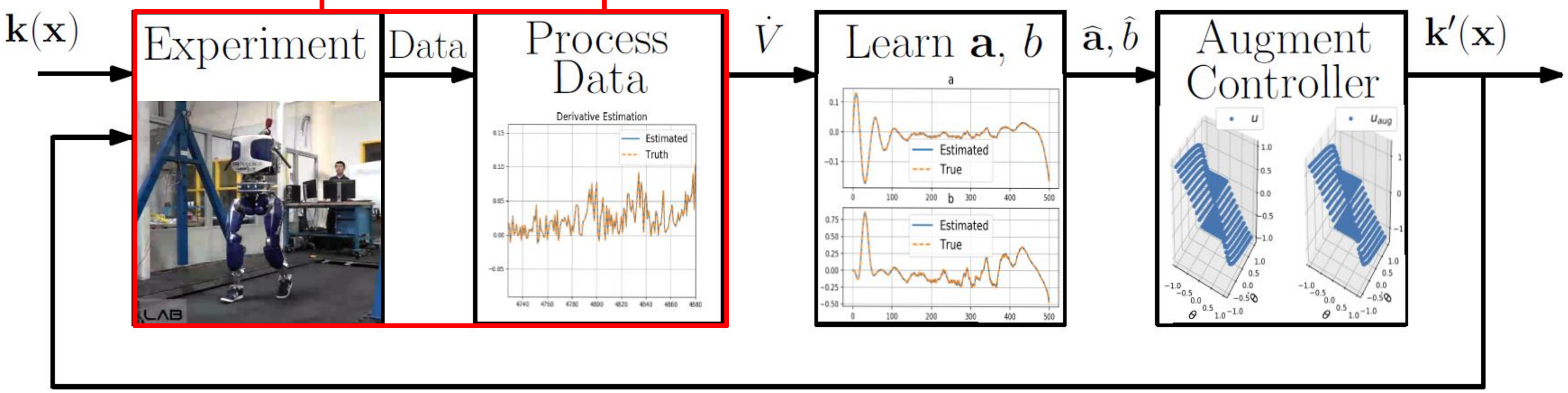
Raw Data

$$D_r = \{\mathbf{x}_i, t_i, \mathbf{u}_i)\}_{i=1}^{N}$$

Processed Data

$$D_p = \{(\mathbf{x}_i, \mathbf{u}_i), \dot{V}_i\}_{i=2}^{N-1}$$



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Learning Control Lyapunov Functions

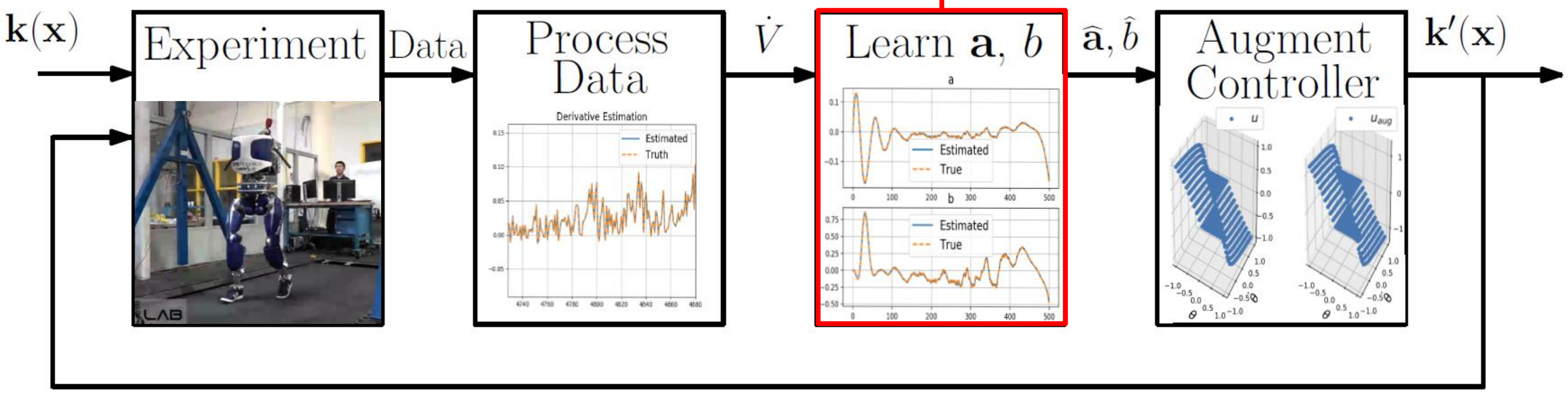
Model Classes

$\widehat{\mathbf{a}} \in \mathcal{H}_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^m$

$\widehat{b} \in \mathcal{H}_b : \mathbb{R}^n \to \mathbb{R}$

Supervised Learning

$$\inf_{\substack{\widehat{\mathbf{a}} \in \mathcal{H}_{\mathbf{a}} \\ \widehat{b} \in \mathcal{H}_b}} \frac{1}{N-2} \sum_{i=2}^{N-1} \mathcal{L}\left(\widehat{\dot{V}}(\mathbf{x}_i, \mathbf{u}_i) + \widehat{\mathbf{a}}(\mathbf{x}_i)^\top \mathbf{u}_i + \widehat{b}(\mathbf{x}_i), \dot{V}_i\right)$$



A. Taylor, Episodic Learning with CLFs for Uncertain Robotic Systems, 2019

Learning Control Lyapunov Functions

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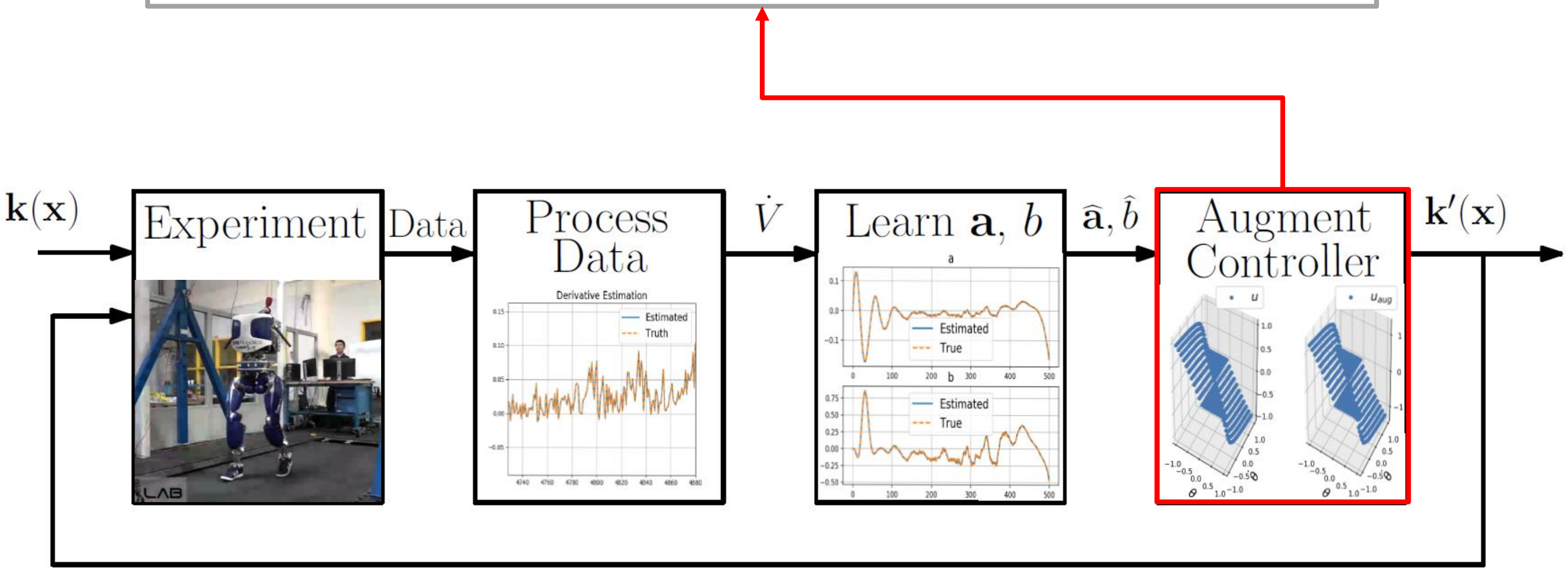
Learning-Augmented Controller

$$\mathbf{k}'(\mathbf{x}) = \underset{\mathbf{u}' \in \mathbb{R}^m}{\operatorname{argmin}} \; \|\mathbf{k}(\mathbf{x}) + \mathbf{u}'\|_2$$
$$\text{s.t. } \widehat{\dot{V}}(\mathbf{x}, \mathbf{k}(\mathbf{x}) + \mathbf{u}') + \widehat{\mathbf{a}}(\mathbf{x})^\top (\mathbf{k}(\mathbf{x}) + \mathbf{u}') + \widehat{b}(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|_2)$$



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Episodic Learning

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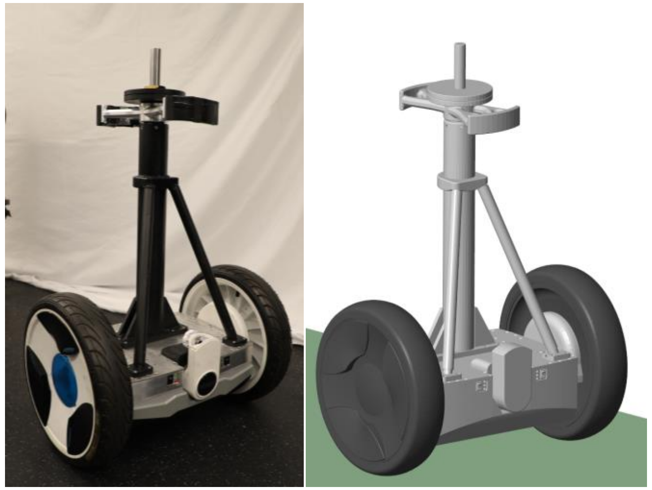
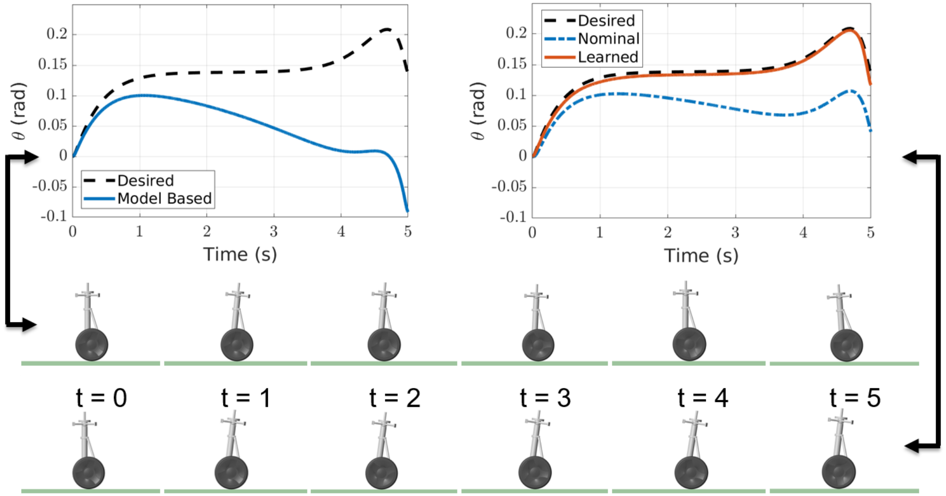
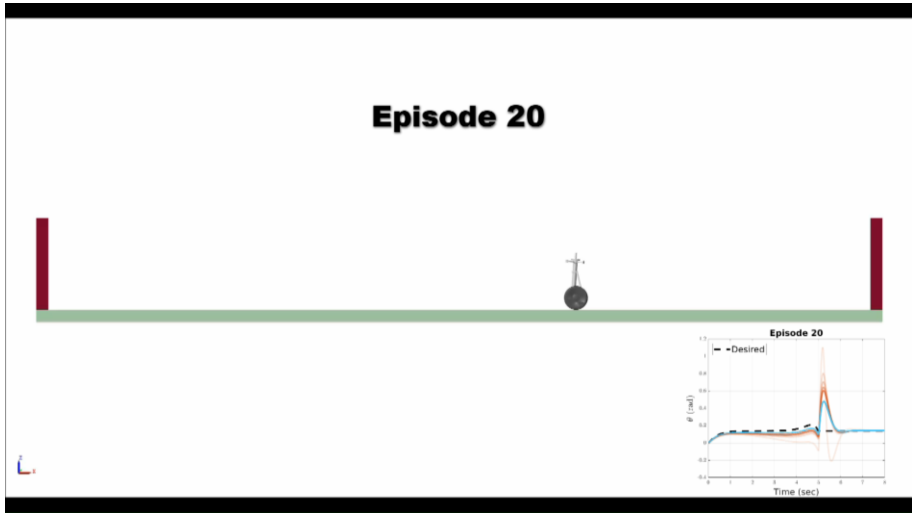


Algorithm 1 Dataset Aggregation for Control Lyapunov Functions (DaCLyF) [31]

Require: Lyapunov function V, Lyapunov function derivative estimate $\hat{\dot{V}}_0$, model classes $\mathcal{H}_{\mathbf{a}}$ and $\mathcal{H}_b$, loss function $\mathcal{L}$, set of initial conditions $\mathcal{X}_0$, nominal state-feedback controller $\mathbf{u}_0$, number of experiments T, sequence of trust coefficients $0 \leq w_1 \leq \cdots \leq w_T \leq 1$

$D = \emptyset$ ▷ Initialize dataset
for $k = 1, \dots, T$ **do**
  $\mathbf{x}_0 \leftarrow \text{sample}(\mathcal{X}_0)$ ▷ Sample initial condition
  $D_k \leftarrow \text{experiment}(\mathbf{x}_0, \mathbf{u}_{k-1})$ ▷ Execute experiment
  $D \leftarrow D \cup D_k$ ▷ Aggregate dataset
  $\hat{\mathbf{a}}, \hat{b} \leftarrow \text{ERM}(\mathcal{H}_{\mathbf{a}}, \mathcal{H}_b, \mathcal{L}, D, \hat{\dot{V}}_0)$ ▷ Fit estimators
  $\hat{\dot{V}}_k \leftarrow \hat{\dot{V}}_0 + \hat{\mathbf{a}}^\top \mathbf{u} + \hat{b}$ ▷ Update derivative estimator
  $\mathbf{u}_k \leftarrow \mathbf{u}_0 + w_k \cdot \text{augment}(\mathbf{u}_0, \hat{\dot{V}}_k)$ ▷ Update controller
end for
return $D, \hat{\dot{V}}_T, \mathbf{u}_T$





S. Ross, et al., A reduction of imitation learning and structured prediction to no-regret online learning.
A. Taylor, Episodic Learning with CLFs for Uncertain Robotic Systems, 2019

Residual Learning Error

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$$\dot{V}(\mathbf{x},\mathbf{u}) \approx \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x})$$

Residual Learning Error

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Residual Learning Error

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$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \mathbf{a}(\mathbf{x})^\top \mathbf{u} + b(\mathbf{x})$$

$$\Downarrow$$

$$\pm \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x})$$

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Caltech

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \mathbf{a}(\mathbf{x})^\top \mathbf{u} + b(\mathbf{x})$$

$$\pm \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x})$$

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) + \underbrace{(\mathbf{a}(\mathbf{x}) - \widehat{\mathbf{a}}(\mathbf{x}))^\top}_{\widetilde{\mathbf{a}}(\mathbf{x})^\top} \mathbf{u} + \underbrace{b(\mathbf{x}) - \widehat{b}(\mathbf{x})}_{\widetilde{b}(\mathbf{x})}$$

Residual Learning Error

Caltech

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \mathbf{a}(\mathbf{x})^\top\mathbf{u} + b(\mathbf{x})$$

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Can we quantify $\widetilde{\mathbf{a}}, \widetilde{b}$?

Residual Learning Error

Caltech

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Can we quantify $\widetilde{\mathbf{a}}, \widetilde{b}$?

If so, what can we say about stability?

Input-to-State Stability

Caltech

Disturbed Dynamics
$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \mathbf{d} \quad (\star)$

Essentially Bounded
$\mathbf{d} \in \mathcal{D}, \ \text{ess. sup.}\{\Vert\mathbf{d}(t)\Vert, t \geq 0\} < \infty$

Input-to-State Stability

Caltech

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$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \mathbf{d} \quad (\star)$$

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Definition 5 (*Input to State Stability*)**.** Under a continuous state-feedback controller $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ the system governed by $(\star)$ is *Input to State Stable* (ISS) if there exist $\beta \in \mathcal{KL}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that it satisfies:

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t) + \gamma\left(\sup_{\tau \geq 0} \|\mathbf{d}(\tau)\|\right),$$

for all $t \geq 0$.

Input-to-State Stability

Caltech

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Definition 6 (*Input to State Stable Control Lyapunov Function*)**.** A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is an *Input to State Stable Control Lyapunov Function* (ISS-CLF) for $(\star)$ on $\mathbb{R}^n$ if there exist $\underline{\alpha}, \overline{\alpha}, \alpha, \rho \in \mathcal{K}_\infty$ such that:

$$\underline{\alpha}(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \overline{\alpha}(\|\mathbf{x}\|)$$
$$\|\mathbf{x}\| \geq \rho(\|\mathbf{d}\|) \implies \inf_{\mathbf{u} \in \mathbb{R}^m} \dot{V}(\mathbf{x}, \mathbf{u}, \mathbf{d}) \leq -\alpha(\|\mathbf{x}\|),$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{d} \in \mathcal{D}$.

Input-to-State Stability

Caltech

Disturbed Dynamics

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \mathbf{d} \quad (\star)$$

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ISS-CLF V for $(\star)$

$\implies$

$\mathbf{k}$ s.t. $(\star)$ is ISS

R. Freeman, P. Kokotovic, Inverse Optimality in robust stabilization, 1996
E. Sontag, Y. Wang, On Characterizations of input-to-state stability with respect to compact sets, 1995

Input-to-State Stability

Caltech

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$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t) + \gamma\left(\sup_{\tau \geq 0} \|\mathbf{d}(\tau)\|\right),$$

for all $t \geq 0$.

Lemma 1. *A sublevel set $\Omega \subseteq \mathcal{X}$ of an ISS-CLF V can be rendered forward invariant, provided $\|\mathbf{x}\| \geq \rho(\|\mathbf{d}\|)$ for all $\mathbf{x} \in \partial\Omega$ and appropriately restricted $\mathbf{d} \in \mathcal{D}$.*

Definition 6 (*Input to State Stable Control Lyapunov Function*). A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is an *Input to State Stable Control Lyapunov Function* (ISS-CLF) for $(\star)$ on $\mathbb{R}^n$ if there exist $\underline{\alpha}, \overline{\alpha}, \alpha, \rho \in \mathcal{K}_\infty$ such that:

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$$\text{ISS-CLF } V \text{ for } (\star) \implies \mathbf{k} \text{ s.t. } (\star) \text{ is ISS}$$

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Projection-to-State Stability

Caltech

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Projection-to-State Stability

Caltech

Definition 8 (*Dynamic Projection*)**.** A continuously differentiable function $\mathbf{\Pi} : \mathbb{R}^n \to \mathbb{R}^k$ is a *dynamic projection* if there exist $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ satisfying:

$$\underline{\sigma}(\|\mathbf{x}\|) \leq \|\mathbf{\Pi}(\mathbf{x})\| \leq \overline{\sigma}(\|\mathbf{x}\|),$$

for all $\mathbf{x} \in \mathbb{R}^n$

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Caltech

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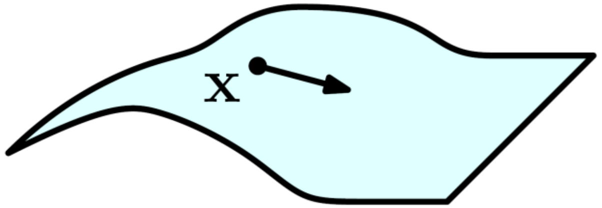

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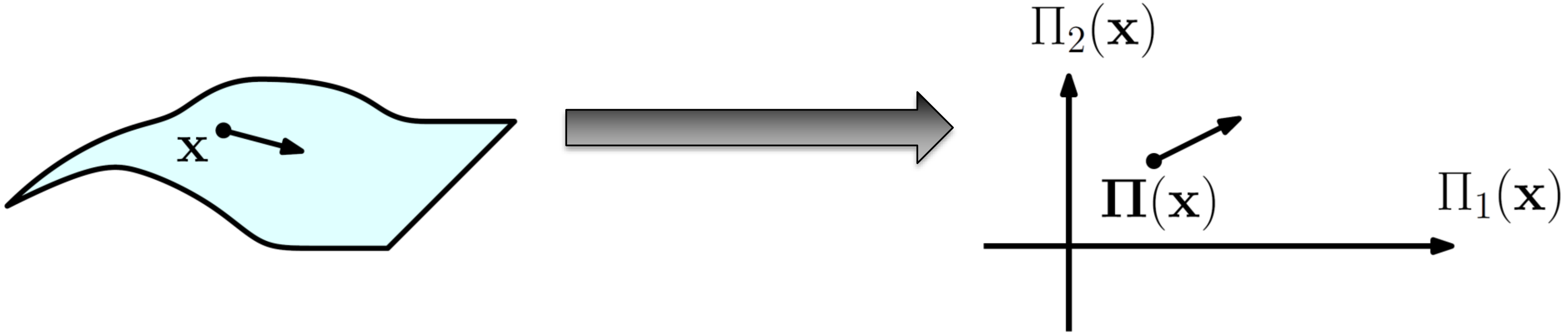

Projected Dynamics

$$\dot{\mathbf{y}} = \mathbf{D}_{\mathbf{\Pi}}(\mathbf{x})\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \underbrace{\mathbf{D}_{\mathbf{\Pi}}(\mathbf{x})\mathbf{d}}_{\delta} \qquad (\star\star)$$

Projection-to-State Stability

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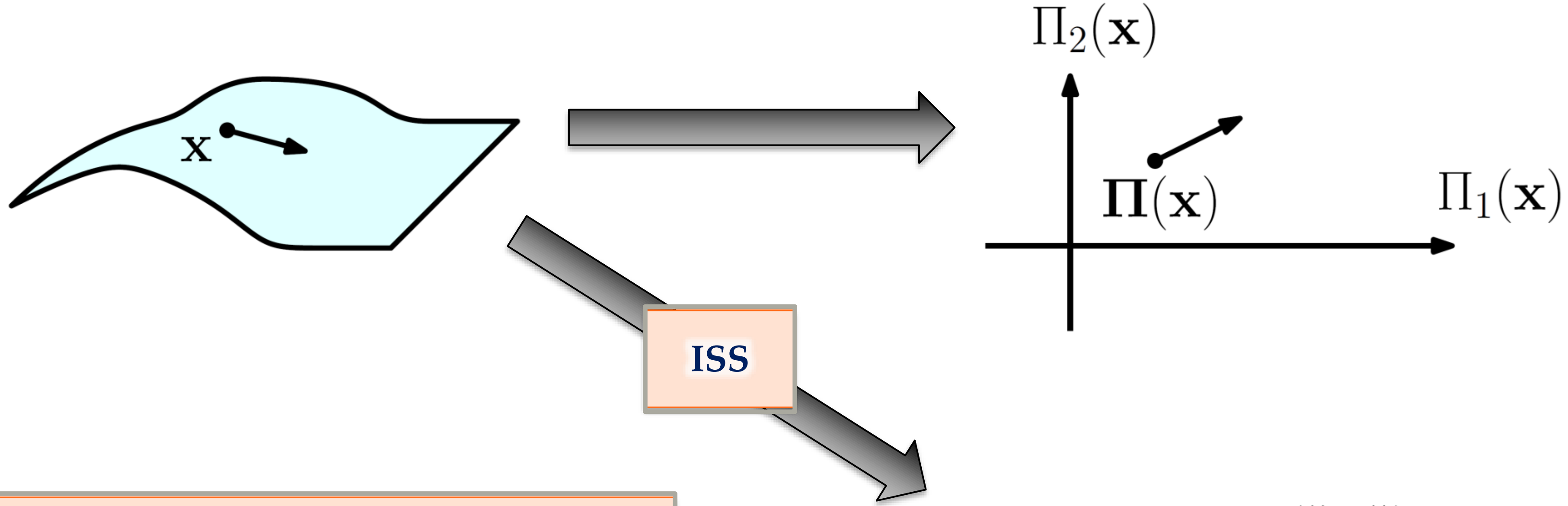

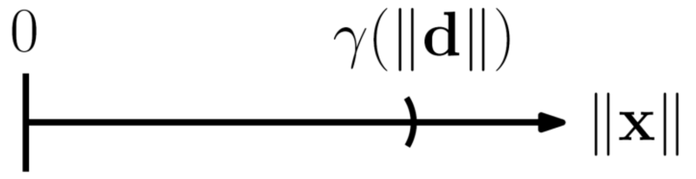

Projected Dynamics

$$\dot{\mathbf{y}} = \mathbf{D}_{\mathbf{\Pi}}(\mathbf{x})\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \underbrace{\mathbf{D}_{\mathbf{\Pi}}(\mathbf{x})\mathbf{d}}_{\delta} \quad (\star\star)$$

Projection-to-State Stability

Caltech

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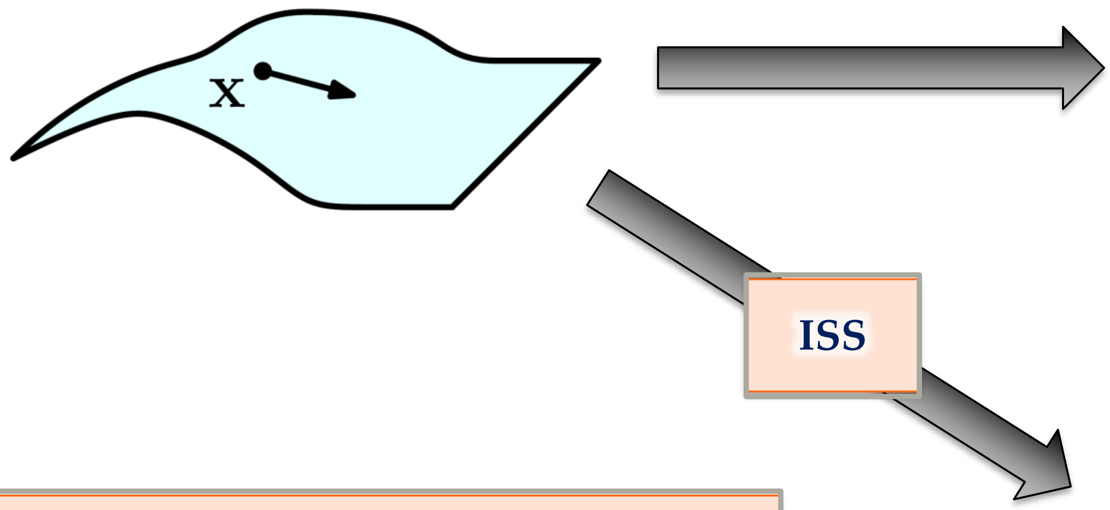

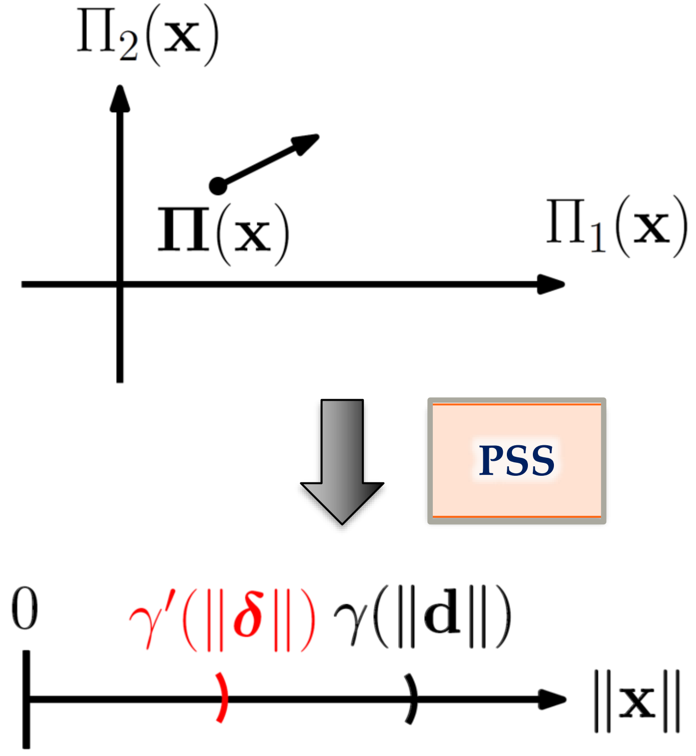

Projected Dynamics

$$\dot{\mathbf{y}} = \mathbf{D_\Pi}(\mathbf{x})\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \underbrace{\mathbf{D_\Pi}(\mathbf{x})\mathbf{d}}_{\delta} \qquad (\star\star)$$

Projection-to-State Stability

Caltech

Definition 8 (*Dynamic Projection*). A continuously differentiable function $\mathbf{\Pi} : \mathbb{R}^n \to \mathbb{R}^k$ is a *dynamic projection* if there exist $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ satisfying:

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Definition 5 (*Input to State Stability*). Under a continuous state-feedback controller $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ the system governed by $(\star)$ is *Input to State Stable* (ISS) if there exist $\beta \in \mathcal{KL}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that it satisfies:

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t) + \gamma\left(\sup_{\tau \geq 0} \|\mathbf{d}(\tau)\|\right),$$

for all $t \geq 0$.

Definition 9 (*Projection to State Stability*). Under a continuous state-feedback controller $\mathbf{k} : \mathbb{R}^n \to \mathcal{U}$, a system is *Projection to State Stable* (PSS) with respect to the dynamic projection $\mathbf{\Pi}$ if there exist $\beta \in \mathcal{KL}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that the solution to $(\star)$ satisfies:

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t) + \gamma\left(\sup_{\tau \geq 0} \|\boldsymbol{\delta}(\tau)\|\right),$$

for all $t \geq 0$, with $\boldsymbol{\delta}$ as defined in $(\star\star)$.

Projected Dynamics

$$\dot{\mathbf{y}} = \mathbf{D}_{\mathbf{\Pi}}(\mathbf{x})\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \underbrace{\mathbf{D}_{\mathbf{\Pi}}(\mathbf{x})\mathbf{d}}_{\boldsymbol{\delta}} \qquad (\star\star)$$

Projection-to-State Stability

Caltech

Disturbed Dynamics

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \mathbf{d} \quad (\star)$$

Projected Dynamics

$$\dot{\mathbf{y}} = \mathbf{D}_\Pi(\mathbf{x}) \left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} \right) + \underbrace{\mathbf{D}_\Pi(\mathbf{x})\mathbf{d}}_{\delta} \quad (\star\star)$$

Projection-to-State Stability

Caltech

Disturbed Dynamics

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \mathbf{d} \qquad (\star)$$

Projected Dynamics

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Theorem 1. *The system governed by* $(\star)$ *can be rendered PSS with respect to the dynamic projection* $\boldsymbol{\Pi}$ *if the system governed by* $(\star\star)$ *has an ISS-CLF satisfying the continuous control property.*

Projection-to-State Stability

Caltech

Disturbed Dynamics

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \mathbf{d} \quad (\star)$$

Projected Dynamics

$$\dot{\mathbf{y}} = \mathbf{D}_{\Pi}(\mathbf{x})\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \underbrace{\mathbf{D}_{\Pi}(\mathbf{x})\mathbf{d}}_{\delta} \quad (\star\star)$$

Theorem 1. *The system governed by* $(\star)$ *can be rendered PSS with respect to the dynamic projection* Π *if the system governed by* $(\star\star)$ *has an ISS-CLF satisfying the continuous control property.*

Corollary 1. *Suppose* $V : \mathbb{R}^n \to \mathbb{R}_+$ *is a CLF satisfying the continuous control property for the undisturbed system* $(\star)$ *(with* $\mathbf{d} \equiv \mathbf{0}$*). Then the disturbed system governed by* $(\star)$ *is PSS with respect to the projection* V*.*

Application of PSS to Learning

Caltech

ISS

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{\mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{b}(\mathbf{x})}_{\mathbf{d}}$$

Application of PSS to Learning

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$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{\mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{b}(\mathbf{x})}_{\mathbf{d}}$$

⇓

PSS

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widehat{b}(\mathbf{x}) + \underbrace{\widetilde{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widetilde{b}(\mathbf{x})}_{\delta}$$

Projected Disturbance

$$\delta = \widetilde{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widetilde{b}(\mathbf{x})$$

Application of PSS to Learning

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ISS

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{\mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{b}(\mathbf{x})}_{\mathbf{d}}$$

PSS

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widehat{b}(\mathbf{x}) + \underbrace{\widetilde{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widetilde{b}(\mathbf{x})}_{\delta}$$

Projected Disturbance

$$\delta = \widetilde{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widetilde{b}(\mathbf{x})$$

Can we characterize δ ?

Uncertainty Functions

Caltech

Definition 10 (*Uncertainty Function*)**.** Let $\mathcal{P}(\mathbb{R}^m \times \mathbb{R})$ denote the set of all subsets of $\mathbb{R}^m \times \mathbb{R}$. An *uncertainty function* is a function $\Delta : \mathcal{X} \to \mathcal{P}(\mathbb{R}^m \times \mathbb{R})$ with $\Delta(\mathbf{x})$ bounded and satisfying $(\widetilde{\mathbf{a}}(\mathbf{x}), \widetilde{b}(\mathbf{x})) \in \Delta(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$.

Uncertainty Functions

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⇩

$$\dot{V}(\mathbf{x}, \mathbf{u}, \delta) \leq \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) + \sup_{(\mathbf{a}, b) \in \Delta(\mathbf{x})} \left(\mathbf{a}^\top \mathbf{u} + b\right)$$

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CLF Estimator Assumption

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|_2)$$

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$$\mathcal{U}(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^m \mid \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|_2)\}$$

Uncertainty Functions to PSS

Caltech

Theorem* *Suppose there is a set* $\mathcal{E}$, $\mathbf{0} \in \mathcal{E}$, *and* $\mu \geq 0$ *satisfying:*

$$\sup_{(\mathbf{a},b)\in\Delta(\mathbf{x})} (\mathbf{a}^\top \mathbf{k}(\mathbf{x}) + b) \leq \mu,$$

for all $\mathbf{x} \in \mathcal{E}$. *Then* $(\star)$ *is PSS with respect to* V.

Uncertainty Functions to PSS

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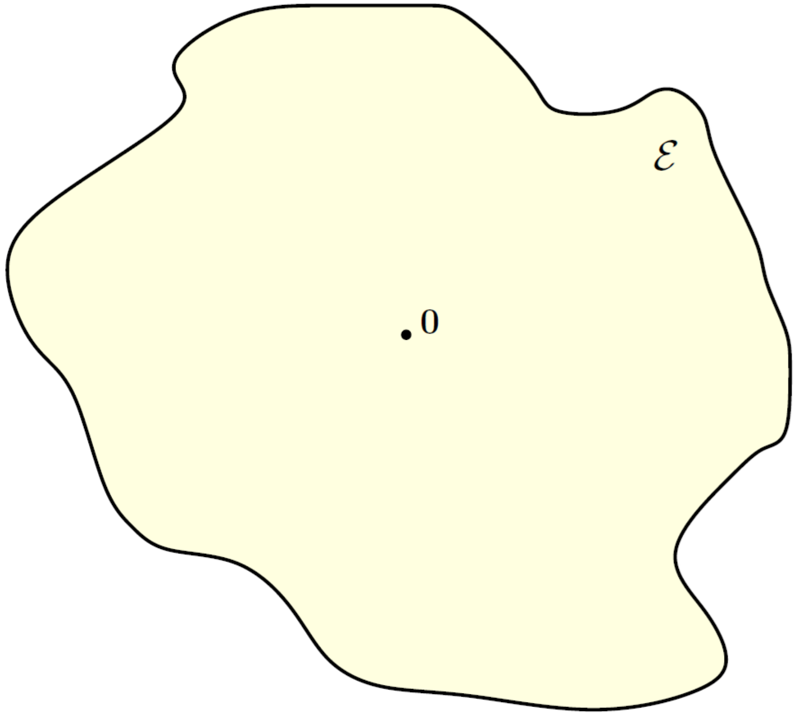

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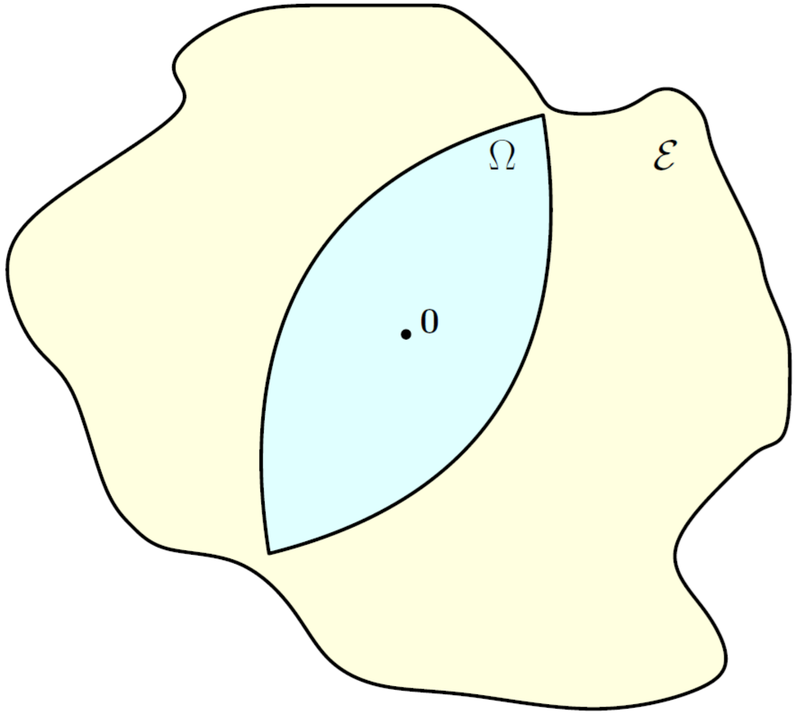

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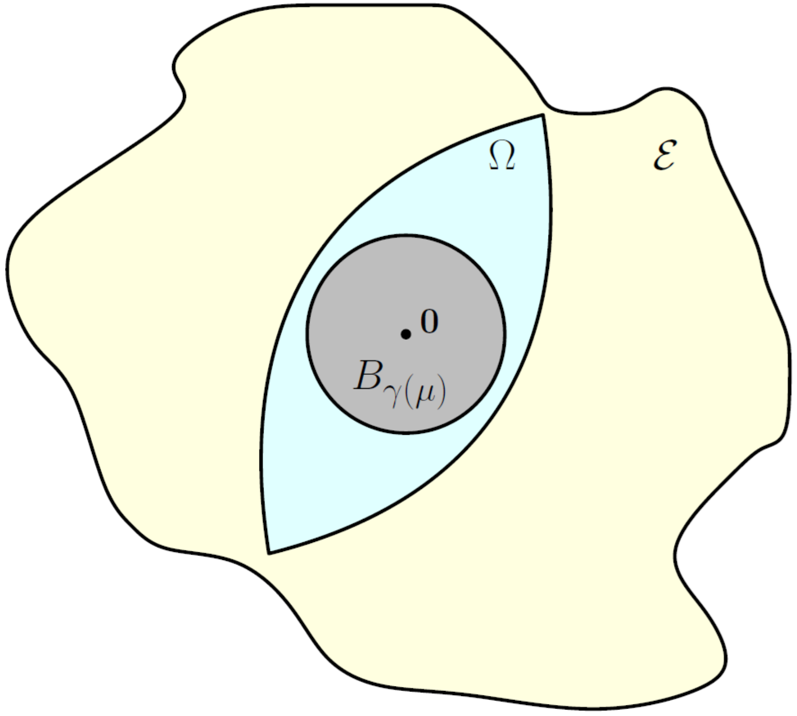

Uncertainty Set Improvement

Caltech

Theorem* *Suppose there is a set* $\mathcal{E}$, $\mathbf{0} \in \mathcal{E}$, *and* $\mu \geq 0$ *satisfying:*

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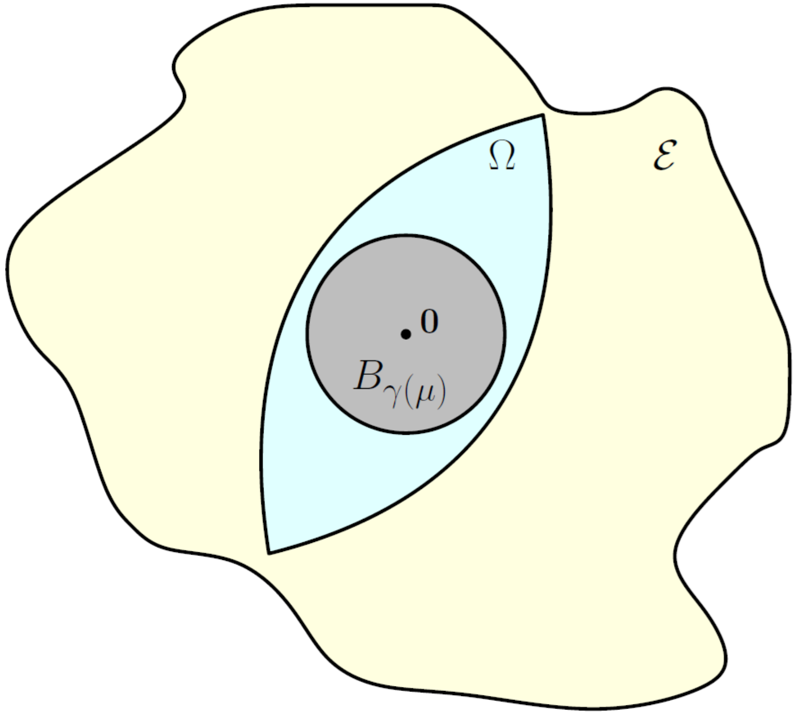
Improvement

$$\Delta'(\mathbf{x}) \subseteq \Delta(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^n$$



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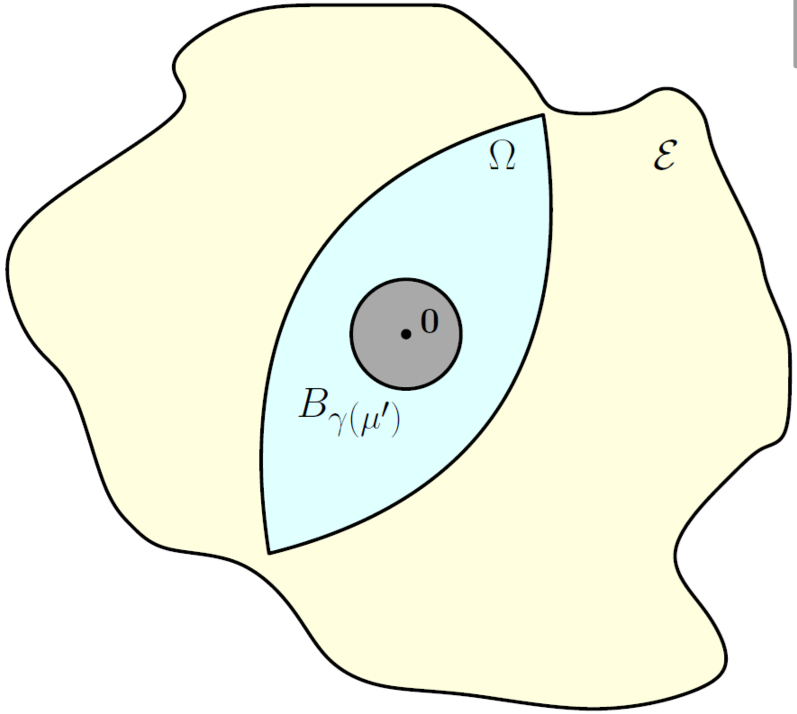

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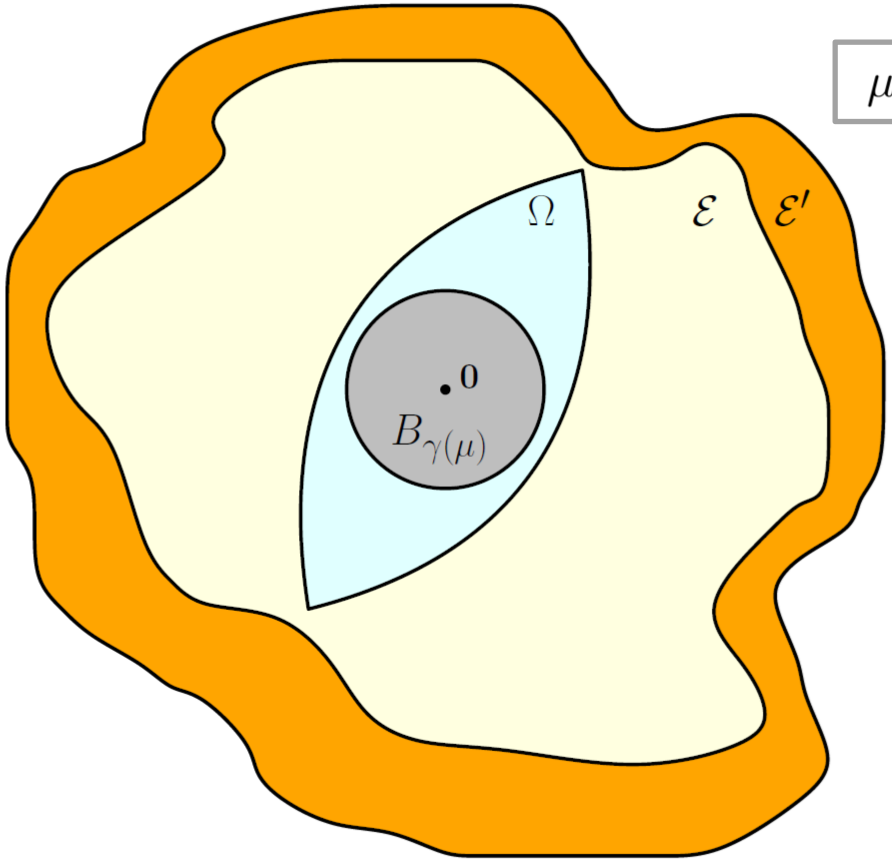

Uncertainty Function Construction

Caltech

Proposition 1. Given a dataset D, an uncertainty function Δ can be constructed as:

$$\Delta(\mathbf{x}) = \{(\mathbf{a}, b) \in \mathbb{R}^m \times \mathbb{R} : \pm(\mathbf{a}^\top \mathbf{u}' + b) \leq \epsilon(\mathbf{x}, \mathbf{x}', \mathbf{u}') \\ \text{for all } (\mathbf{x}', \mathbf{u}') \in D_0\},$$

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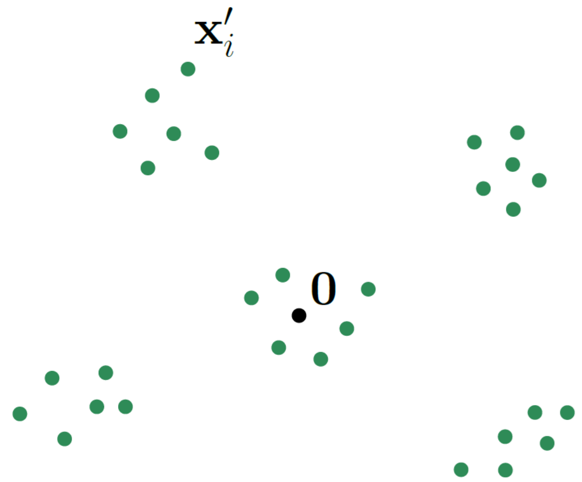

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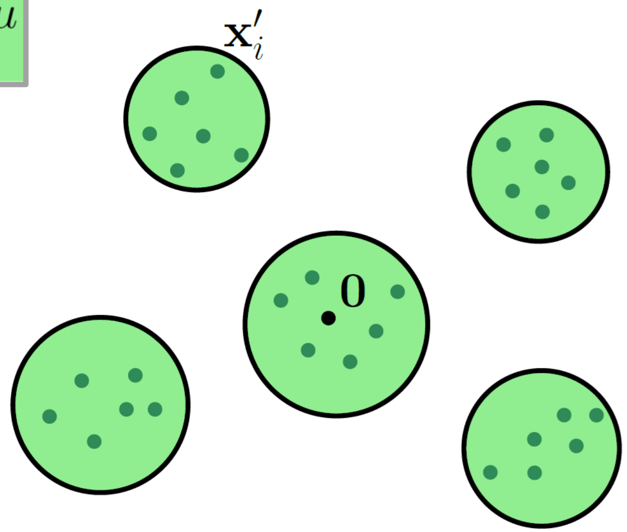

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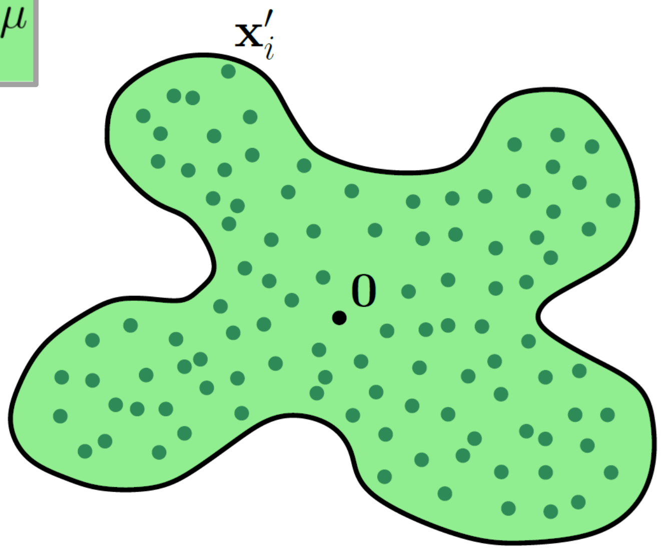

Simulation Results

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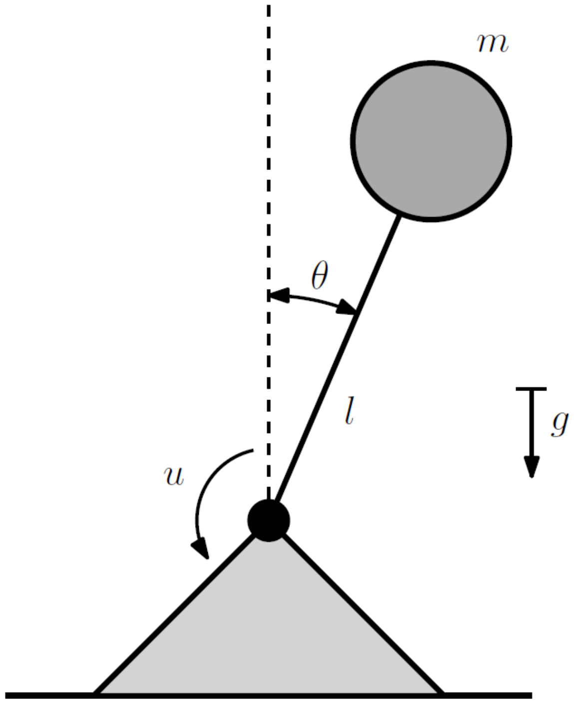

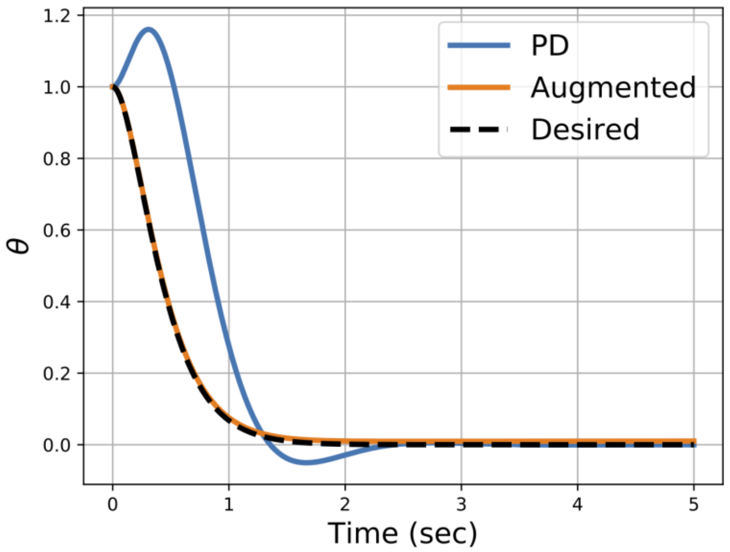

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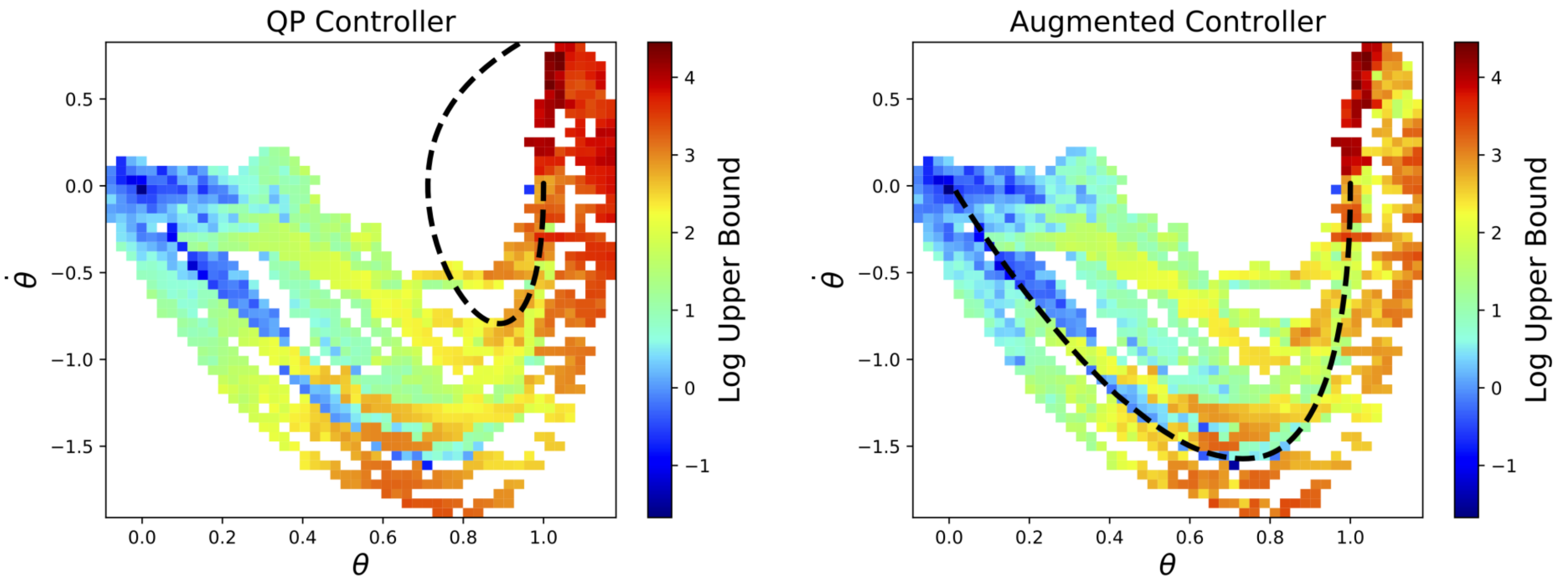

Conclusions

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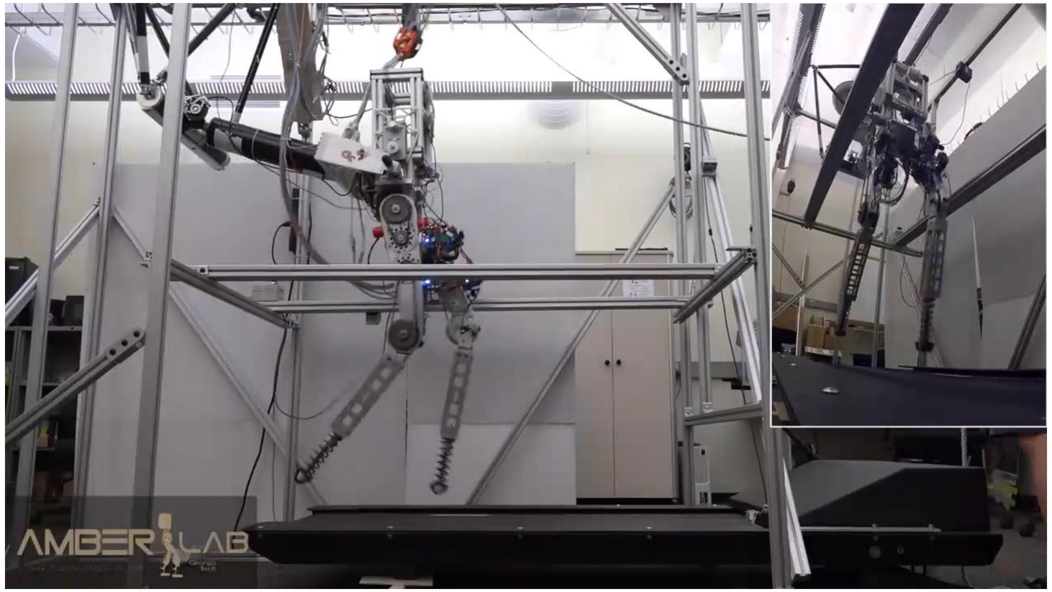
- **Projection-to-State Stability** offers alternative approach for studying stability with disturbances
- Data driven methods to bound residual uncertainty after learning
- Gap between data necessary for good performance and certifying stability.



Conclusions

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Thank You!

A Control Lyapunov Perspective on Episodic Learning via Projection to State Stability

Andrew Taylor Victor Dorobantu Meera Krishnamoorthy
Hoang Le Yisong Yue Aaron Ames

Removed Slides

Class - $\mathcal{K}$ Functions

Definition 1 (*Class $\mathcal{K}$ Function*)**.** A continuous function $\alpha : [0, a) \to \mathbb{R}_+$, with $a > 0$, is *class* $\mathcal{K}$, denoted $\alpha \in \mathcal{K}$, if it is monotonically (strictly) increasing and satisfies $\alpha(0) = 0$. If the domain of α is all of $\mathbb{R}_+$ and $\lim_{r\to\infty} \alpha(r) = \infty$, then α is termed radially unbounded and *class* $\mathcal{K}_\infty$.

Definition 2 (*Class $\mathcal{KL}$ Function*)**.** A continuous function $\beta : [0, a) \times \mathbb{R}_+ \to \mathbb{R}_+$, with $a > 0$, is *class* $\mathcal{KL}$, denoted $\beta \in \mathcal{KL}$, if the function $r \mapsto \beta(r, s) \in \mathcal{K}$ for all $s \in \mathbb{R}_+$, and the function $s \mapsto \beta(r, s)$ is monotonically non-increasing with $\beta(r, s) \to 0$ as $s \to \infty$ for all $r \in [0, a)$.

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Inverses & Composition

$$\alpha \in \mathcal{K}(\mathcal{K}_\infty) \implies \alpha^{-1} : [0, \alpha(a)) \to \mathbb{R}_+ \in \mathcal{K}(\mathcal{K}_\infty)$$

$$\alpha, \gamma \in \mathcal{K}_\infty \implies \gamma \circ \alpha : [0, \infty) \to \mathbb{R}_+ \in \mathcal{K}_\infty$$

Projection-to-State Stability

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widehat{b}(\mathbf{x}) + \underbrace{(\mathbf{a}(\mathbf{x}) - \widehat{\mathbf{a}}(\mathbf{x}))^\top}_{\widetilde{\mathbf{a}}(\mathbf{x})^\top}\mathbf{u} + \underbrace{b(\mathbf{x}) - \widehat{b}(\mathbf{x})}_{\widetilde{b}(\mathbf{x})}$$

Projection-to-State Stability

Definition 8 (*Dynamic Projection*)**.** A continuously differentiable function $\mathbf{\Pi} : \mathbb{R}^n \to \mathbb{R}^k$ is a *dynamic projection* if there exist $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ satisfying:

$$\underline{\sigma}(\|\mathbf{x}\|) \leq \|\mathbf{\Pi}(\mathbf{x})\| \leq \overline{\sigma}(\|\mathbf{x}\|),$$

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$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \mathbf{d} \quad (\star)$$

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$\mathbf{y} = \mathbf{\Pi}(\mathbf{x})$ →

Projected Dynamics

$$\dot{\mathbf{y}} = \mathbf{D}_{\mathbf{\Pi}}(\mathbf{x}) \left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \underbrace{\mathbf{D}_{\mathbf{\Pi}}(\mathbf{x})\mathbf{d}}_{\delta} \quad (\star\star)$$

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Essentially Bounded

$$\text{ess. sup.}\{\|\boldsymbol{\delta}(t)\|, t \geq 0\} < \infty$$

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Definition 9 (*Projection to State Stability*). Under a continuous state-feedback controller $\mathbf{k} : \mathbb{R}^n \to \mathcal{U}$, a system is *Projection to State Stable* (PSS) with respect to the dynamic projection $\mathbf{\Pi}$ if there exist $\beta \in \mathcal{KL}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that the solution to $(\star)$ satisfies:

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t) + \gamma\left(\sup_{\tau \geq 0} \|\boldsymbol{\delta}(\tau)\|\right),$$

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Definition 5 (*Input to State Stability*)**.** Under a continuous state-feedback controller $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ the system governed by $(\star)$ is *Input to State Stable* (ISS) if there exist $\beta \in \mathcal{KL}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that it satisfies:

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t) + \gamma\left(\sup_{\tau \geq 0} \|\mathbf{d}(\tau)\|\right),$$

for all $t \geq 0$.

Definition 9 (*Projection to State Stability*)**.** Under a continuous state-feedback controller $\mathbf{k} : \mathbb{R}^n \to \mathcal{U}$, a system is *Projection to State Stable* (PSS) with respect to the dynamic projection $\mathbf{\Pi}$ if there exist $\beta \in \mathcal{KL}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that the solution to $(\star)$ satisfies:

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$$\ldots_{\Pi}(\mathbf{x})\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \underbrace{\mathbf{D}_{\Pi}(\mathbf{x})\mathbf{d}}_{\boldsymbol{\delta}} \qquad (\star\star)$$

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Projection-to-State Stability

Theorem 1. *The system governed by* ($\star$) *can be rendered PSS with respect to the dynamic projection* $\mathbf{\Pi}$ *if the system governed by* ($\star\star$) *has an ISS-CLF satisfying the continuous control property.*

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$$\underline{\sigma}(\|\mathbf{x}\|) \leq \|\mathbf{\Pi}(\mathbf{x})\| \leq \overline{\sigma}(\|\mathbf{x}\|)$$

$$\alpha(a+b) \leq \alpha(2a) + \alpha(2b)$$

PSS

$$\|\mathbf{x}(t)\| \leq \beta'(\|\mathbf{x}(0)\|, t) + \gamma'\left(\sup_{\tau\geq 0}\|\boldsymbol{\delta}(\tau)\|\right)$$

C. Kellet, A compendium of comparison function results, 2014

Projection-to-State Stability

Corollary 1. *Suppose* $V : \mathbb{R}^n \to \mathbb{R}_+$ *is a CLF satisfying the continuous control property for the undisturbed system* $(\star)$ *(with* $\mathbf{d} \equiv \mathbf{0}$*). Then the disturbed system governed by* $(\star)$ *is PSS with respect to the projection* V.

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→

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→

$$V(\mathbf{x}) \geq \alpha_2 \circ \alpha_q^{-1}(|\delta|) \implies \inf_{\mathbf{u}\in\mathbb{R}^m} \dot{V}(\mathbf{x},\mathbf{u},\delta) \leq -\alpha_p \circ \alpha_2^{-1}(V(\mathbf{x}))$$

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$$\dot{V}(\mathbf{x}, \mathbf{u}, \delta) = \frac{\partial V}{\partial \mathbf{x}}\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \delta$$

→

$$\dot{V}(\mathbf{x}, \mathbf{k}(\mathbf{x}), \delta) \leq -\alpha_3(\|\mathbf{x}\|_2) + |\delta|$$

↓

$$\dot{V}(\mathbf{x}, \mathbf{k}(\mathbf{x}), \delta) \leq -\alpha_p(\|\mathbf{x}\|_2) - \alpha_q(\|\mathbf{x}\|_2) + |\delta|$$

↓

$$\|\mathbf{x}\|_2 \geq \alpha_q^{-1}(|\delta|) \implies \dot{V}(\mathbf{x}, \mathbf{k}(\mathbf{x}), \delta) \leq -\alpha_p(\|\mathbf{x}\|_2)$$

↓

$$V(\mathbf{x}) \geq \alpha_2 \circ \alpha_q^{-1}(|\delta|) \implies \inf_{\mathbf{u} \in \mathbb{R}^m} \dot{V}(\mathbf{x}, \mathbf{u}, \delta) \leq -\alpha_p \circ \alpha_2^{-1}(V(\mathbf{x}))$$

Identity Map is an ISS-CLF!

From ISS to PSS

ISS
$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{\mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{b}(\mathbf{x})}_{\mathbf{d}}$

From ISS to PSS

ISS

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{\mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{b}(\mathbf{x})}_{\mathbf{d}}$$

⇓

PSS

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widehat{b}(\mathbf{x}) + \underbrace{\widetilde{\mathbf{a}}(\mathbf{x})\mathbf{u} + \widetilde{b}(\mathbf{x})}_{\delta}$$

From ISS to PSS

ISS

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{\mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{b}(\mathbf{x})}_{\mathbf{d}}$$

PSS

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Projected Disturbance

$$\delta = \widetilde{\mathbf{a}}(\mathbf{x})\mathbf{u} + \widetilde{b}(\mathbf{x})$$

From ISS to PSS

ISS

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \underbrace{\mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{b}(\mathbf{x})}_{\mathbf{d}}$$

⇓

PSS

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) + \underbrace{\widetilde{\mathbf{a}}(\mathbf{x})\mathbf{u} + \widetilde{b}(\mathbf{x})}_{\delta}$$

Projected Disturbance

$$\delta = \widetilde{\mathbf{a}}(\mathbf{x})\mathbf{u} + \widetilde{b}(\mathbf{x})$$

Can we characterize δ ?

Uncertainty Functions

Definition 10 (*Uncertainty Function*)**.** Let $\mathcal{P}(\mathbb{R}^m \times \mathbb{R})$ denote the set of all subsets of $\mathbb{R}^m \times \mathbb{R}$. An *uncertainty function* is a function $\Delta : \mathcal{X} \to \mathcal{P}(\mathbb{R}^m \times \mathbb{R})$ with $\Delta(\mathbf{x})$ bounded and satisfying $(\widetilde{\mathbf{a}}(\mathbf{x}), \widetilde{b}(\mathbf{x})) \in \Delta(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$.

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$$\dot{V}(\mathbf{x}, \mathbf{u}, \delta) \leq \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) + \sup_{(\mathbf{a}, b) \in \Delta(\mathbf{x})} \left(\mathbf{a}^\top \mathbf{u} + b\right)$$

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CLF Estimator Assumption

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|_2)$$

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$$\mathcal{U}(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^m \mid \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|_2)\}$$

Uncertainty Functions

Definition 10 (*Uncertainty Function*). Let $\mathcal{P}(\mathbb{R}^m \times \mathbb{R})$ denote the set of all subsets of $\mathbb{R}^m \times \mathbb{R}$. An *uncertainty function* is a function $\Delta : \mathcal{X} \to \mathcal{P}(\mathbb{R}^m \times \mathbb{R})$ with $\Delta(\mathbf{x})$ bounded and satisfying $(\widetilde{\mathbf{a}}(\mathbf{x}), \widetilde{b}(\mathbf{x})) \in \Delta(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$.

$$\dot{V}(\mathbf{x}, \mathbf{u}, \delta) \leq \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) + \sup_{(\mathbf{a}, b) \in \Delta(\mathbf{x})} \left(\mathbf{a}^\top \mathbf{u} + b\right)$$

CLF Estimator Assumption

$$\inf_{\mathbf{u} \in \mathbb{R}^m} \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|_2)$$

$$\mathcal{U}(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^m \mid \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|_2)\}$$

$$\inf_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \dot{V}(\mathbf{x}, \mathbf{u}, \delta) - \sup_{(\mathbf{a}, b) \in \Delta(\mathbf{x})} \left(\mathbf{a}^\top \mathbf{u} + b\right) \leq -\alpha_3(\|\mathbf{x}\|_2)$$

Uncertainty Functions to PSS

Uncertainty Functions to PSS

Theorem 2 (*Sufficient Conditions for PSS in Affine Control Systems*). *Consider the system in* $(*)$, *and a CLF* V *for* $(**)$ *with estimated time-derivative as defined in* $(***)$ *satisfying the CLF assumption. Let* Δ *be an uncertainty function and let* $\mathbf{k} : \mathbb{R}^n \to \mathcal{U}$ *be a continuous state-feedback controller satisfying* $\mathbf{k}(\mathbf{x}) \in \mathcal{U}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^n$. *Suppose there exists* $\alpha_p, \alpha_q \in \mathcal{K}_\infty$ *with* $\alpha_p + \alpha_q = \alpha_3$ *and a sublevel set* Ω *of* V *satisfying:*

$$\|\mathbf{x}\| \geq \sup_{(\mathbf{a},b)\in\Delta(\mathbf{x})} \alpha_q^{-1}(\mathbf{a}^\top \mathbf{k}(\mathbf{x}) + b),$$

for all $\mathbf{x} \in \partial\Omega$. *Then the system governed by* $(*)$ *is PSS with respect to the projection* V *on* Ω.

$$\dot{\mathbf{x}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} + \mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{b}(\mathbf{x}) \quad (*)$$

$$\widehat{\dot{\mathbf{x}}} = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u} \quad (**)$$

$$\widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top \mathbf{u} + \widehat{b}(\mathbf{x}) \quad (***)$$

Uncertainty Functions to PSS

Caltech

Corollary 2. *Suppose there is a set $\mathcal{E}$ and $\mu \geq 0$ satisfying:*

$$\sup_{(\mathbf{a},b)\in\Delta(\mathbf{x})} (\mathbf{a}^\top \mathbf{k}(\mathbf{x}) + b) \leq \mu,$$

for all $\mathbf{x} \in \mathcal{E}$. If there exists a sublevel set Ω of V such that:

$$B_{\alpha_q^{-1}(\mu)} \subseteq \Omega \subseteq \mathcal{E},$$

then the system is PSS with respect to the (CLF) projection V on Ω, and the smallest sublevel set of V containing $B_{\alpha_q^{-1}(\mu)}$ is asymptotically stable.

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Uncertainty Functions to PSS

Caltech

Corollary 2. *Suppose there is a set $\mathcal{E}$ and $\mu \geq 0$ satisfying:*

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then the system is PSS with respect to the (CLF) projection V on Ω, and the smallest sublevel set of V containing $B_{\alpha_q^{-1}(\mu)}$ is asymptotically stable.

Corollary 3 (*Uncertainty Function Improvement*)**.** *Consider uncertainty functions Δ and Δ', as well as $\mathcal{E}$ and μ as defined in Corollary 2.*

- *Fix $\mu > 0$ and let $\mathcal{E}_\mu$ be defined as:*

$$\mathcal{E}_\mu = \{\mathbf{x} \in \mathcal{X} : \sup_{(\mathbf{a},b)\in\Delta(\mathbf{x})} (\mathbf{a}^\top \mathbf{k}(\mathbf{x}) + b) \leq \mu\}.$$

- *Fix $\mathcal{E} \subseteq \mathcal{X}$ and let $\mu_{\mathcal{E}}$ be defined as:*

$$\mu_{\mathcal{E}} = \sup_{\mathbf{x}\in\mathcal{E}} \sup_{(\mathbf{a},b)\in\Delta(\mathbf{x})} (\mathbf{a}^\top \mathbf{k}(\mathbf{x}) + b).$$

Suppose $\Delta'(\mathbf{x}) \subseteq \Delta(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$. Then the associated set $\mathcal{E}'_\mu$ and scalar $\mu'_{\mathcal{E}}$ satisfy $\mathcal{E}_\mu \subseteq \mathcal{E}'_\mu$ and $\mu'_{\mathcal{E}} \leq \mu_{\mathcal{E}}$.

Uncertainty Function Construction

Caltech

Proposition 1. Given a dataset D, an uncertainty function Δ can be constructed as:

$$\Delta(\mathbf{x}) = \{(\mathbf{a}, b) \in \mathbb{R}^m \times \mathbb{R} : \pm(\mathbf{a}^\top \mathbf{u}' + b) \leq \epsilon(\mathbf{x}, \mathbf{x}', \mathbf{u}') \text{ for all } (\mathbf{x}', \mathbf{u}') \in D_0\},$$

for all $\mathbf{x} \in \mathcal{X}$, where $\epsilon : \mathcal{X} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}_+$ is continuous.

$$\ell(\mathbf{x}, \mathbf{u}) = \left| \dot{V}(\mathbf{x}, \mathbf{u}, \delta) - \widehat{\dot{V}}(\mathbf{x}, \mathbf{u}) \right|$$

Uncertainty Function Construction

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Proposition 1. Given a dataset D, an uncertainty function Δ can be constructed as:

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$$\ell(\mathbf{x}, \mathbf{u}) = \left|\dot{V}(\mathbf{x}, \mathbf{u}, \delta) - \widehat{\dot{V}}(\mathbf{x}, \mathbf{u})\right|$$

⇒

$$|\mathbf{a}(\mathbf{x})^\top \mathbf{u}' + b(\mathbf{x})| \leq \ell(\mathbf{x}', \mathbf{u}') + \epsilon_L(\mathbf{x}, \mathbf{x}')(L_{\mathbf{A}} \|\mathbf{u}'\|_2 + L_{\mathbf{b}}) + \epsilon_\infty(\mathbf{x}, \mathbf{x}')(\|\mathbf{A}\|_\infty \|\mathbf{u}'\| + \|\mathbf{b}\|_\infty).$$

$$\epsilon_L(\mathbf{x}, \mathbf{x}') = \|\mathbf{x} - \mathbf{x}'\| \min\{\|\nabla V(\mathbf{x})\|_2, \|\nabla V(\mathbf{x}')\|_2\}$$

$$\epsilon_\infty(\mathbf{x}, \mathbf{x}') = \|\nabla V(\mathbf{x}) - \nabla V(\mathbf{x}')\|_2$$

$$\epsilon_{\mathcal{H}}(\mathbf{x}, \mathbf{x}', \mathbf{u}') = |(\widehat{\mathbf{a}}(\mathbf{x}) - \widehat{\mathbf{a}}(\mathbf{x}'))^\top \mathbf{u}' + \widehat{b}(\mathbf{x}) - \widehat{b}(\mathbf{x}')|$$

Projection-to-State Stability

Caltech

$$\dot{V}(\mathbf{x},\mathbf{u}) = \widehat{\dot{V}}(\mathbf{x},\mathbf{u}) + \widehat{\mathbf{a}}(\mathbf{x})^\top\mathbf{u} + \widehat{b}(\mathbf{x}) + \underbrace{(\mathbf{a}(\mathbf{x}) - \widehat{\mathbf{a}}(\mathbf{x}))^\top}_{\widetilde{\mathbf{a}}(\mathbf{x})^\top}\mathbf{u} + \underbrace{b(\mathbf{x}) - \widehat{b}(\mathbf{x})}_{\widetilde{b}(\mathbf{x})}$$

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Projection-to-State Stability

Caltech

Definition 8 (*Dynamic Projection*)**.** A continuously differentiable function $\mathbf{\Pi} : \mathbb{R}^n \to \mathbb{R}^k$ is a *dynamic projection* if there exist $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ satisfying:

$$\underline{\sigma}(\|\mathbf{x}\|) \leq \|\mathbf{\Pi}(\mathbf{x})\| \leq \overline{\sigma}(\|\mathbf{x}\|),$$

for all $\mathbf{x} \in \mathbb{R}^n$

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Projection-to-State Stability

Caltech

Definition 8 (*Dynamic Projection*). A continuously differentiable function $\mathbf{\Pi} : \mathbb{R}^n \to \mathbb{R}^k$ is a *dynamic projection* if there exist $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ satisfying:

$$\underline{\sigma}(\|\mathbf{x}\|) \leq \|\mathbf{\Pi}(\mathbf{x})\| \leq \overline{\sigma}(\|\mathbf{x}\|),$$

for all $\mathbf{x} \in \mathbb{R}^n$

Definition 5 (*Input to State Stability*). Under a continuous state-feedback controller $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ the system governed by $(\star)$ is *Input to State Stable* (ISS) if there exist $\beta \in \mathcal{KL}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that it satisfies:

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t) + \gamma\left(\sup_{\tau \geq 0} \|\mathbf{d}(\tau)\|\right),$$

for all $t \geq 0$.

Projected Dynamics

$$\dot{\mathbf{y}} = \mathbf{D}_\Pi(\mathbf{x})\left(\widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})\mathbf{u}\right) + \underbrace{\mathbf{D}_\Pi(\mathbf{x})\mathbf{d}}_{\boldsymbol{\delta}} \quad (\star\star)$$

Definition 9 (*Projection to State Stability*). Under a continuous state-feedback controller $\mathbf{k} : \mathbb{R}^n \to \mathcal{U}$, a system is *Projection to State Stable* (PSS) with respect to the dynamic projection $\mathbf{\Pi}$ if there exist $\beta \in \mathcal{KL}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that the solution to $(\star)$ satisfies:

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t) + \gamma\left(\sup_{\tau \geq 0} \|\boldsymbol{\delta}(\tau)\|\right),$$

for all $t \geq 0$, with $\boldsymbol{\delta}$ as defined in $(\star\star)$.

IROS 2019 Backups

Computed Torque

$$\dot{\boldsymbol{\eta}} = \widehat{\mathbf{f}}(\mathbf{q},\dot{\mathbf{q}}) - \dot{\mathbf{r}}(t) + \widehat{\mathbf{g}}(\mathbf{q})\mathbf{u}$$

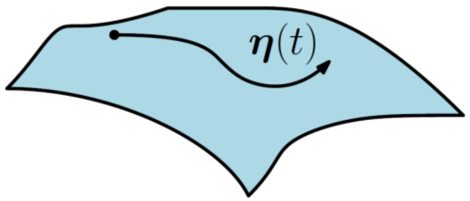
Computed Torque (Feedback Linearization)

$$\mathbf{k}(\mathbf{q},\dot{\mathbf{q}},t) = \widehat{\mathbf{B}}^{-1}\widehat{\mathbf{D}}(\mathbf{q})\left((\widehat{\mathbf{D}}(\mathbf{q}))^{-1}\widehat{\mathbf{H}}(\mathbf{q},\dot{\mathbf{q}}) + \ddot{\mathbf{q}}_d(t) + \boldsymbol{\nu}(\mathbf{q},\dot{\mathbf{q}},t)\right)$$

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{\eta} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\boldsymbol{\nu}(\mathbf{q},\dot{\mathbf{q}},t)$$

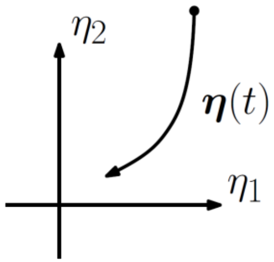
Error PD

$$\boldsymbol{\nu}(\mathbf{q},\dot{\mathbf{q}},t) = -\begin{bmatrix}\mathbf{K}_p & \mathbf{K}_d\end{bmatrix}\boldsymbol{\eta}$$



$$\dot{\boldsymbol{\eta}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\mathbf{K}_p & -\mathbf{K}_d \end{bmatrix}}_{\mathbf{A}_{cl}}\boldsymbol{\eta}$$

$$\mathfrak{Re}(\lambda_i(\mathbf{A}_{cl})) < 0$$
$$i = 1, \dots, n$$



Projection -to-State-Stability (PSS)

- Appearing at CDC 2019:

$$\dot{V}(\boldsymbol{\eta},\mathbf{u}) = \underbrace{\frac{\partial V}{\partial \boldsymbol{\eta}}\left(\widehat{\mathbf{f}}(\mathbf{q},\dot{\mathbf{q}}) - \dot{\mathbf{r}}(t) + \widehat{\mathbf{g}}(\mathbf{q})\mathbf{u}\right)}_{\widehat{\dot{V}}(\boldsymbol{\eta},\mathbf{u})} + \underbrace{\frac{\partial V}{\partial \boldsymbol{\eta}}\mathbf{A}(\mathbf{q})}_{\mathbf{a}(\boldsymbol{\eta},\mathbf{q})^\top}\mathbf{u} + \underbrace{\frac{\partial V}{\partial \boldsymbol{\eta}}\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})}_{b(\boldsymbol{\eta},\mathbf{q},\dot{\mathbf{q}})}$$

Supervised Learning

$$\begin{aligned}\dot{V}(\boldsymbol{\eta},\mathbf{u}) = \widehat{\dot{V}}(\boldsymbol{\eta},\mathbf{u}) + \widehat{\mathbf{a}}(\boldsymbol{\eta},\mathbf{q})^\top\mathbf{u} + \widehat{b}(\boldsymbol{\eta},\mathbf{q},\dot{\mathbf{q}}) \\ + \underbrace{\left(\mathbf{a}(\boldsymbol{\eta},\mathbf{q}) - \widehat{\mathbf{a}}(\boldsymbol{\eta},\mathbf{q})\right)^\top\mathbf{u} + b(\boldsymbol{\eta},\mathbf{q},\dot{\mathbf{q}}) - \widehat{b}(\boldsymbol{\eta},\mathbf{q},\dot{\mathbf{q}})}_{\delta(\boldsymbol{\eta},\mathbf{q},\dot{\mathbf{q}},\mathbf{u})}\end{aligned}$$

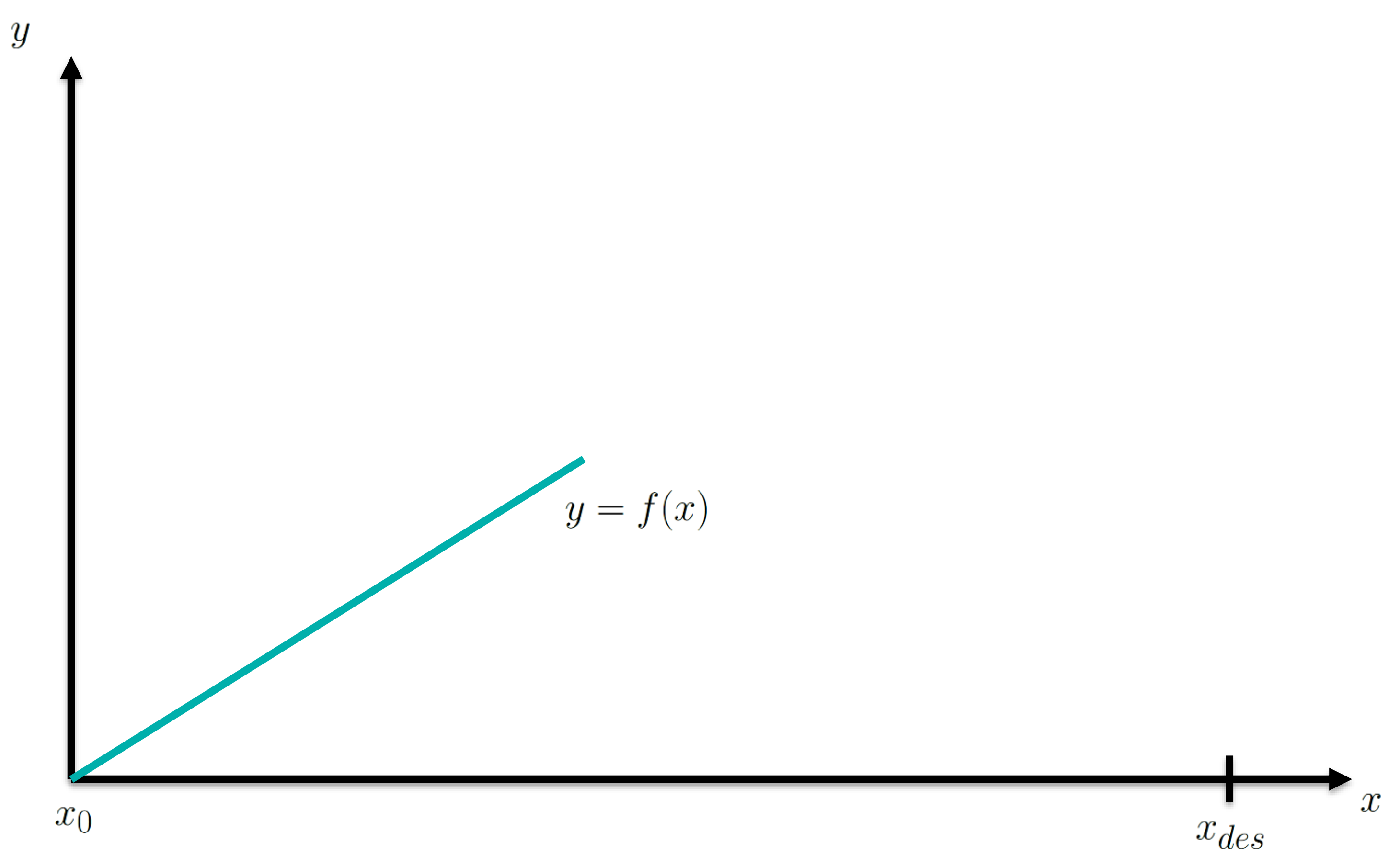
PSS

$$\|\boldsymbol{\eta}(t)\| \leq \beta(\|\boldsymbol{\eta}(0)\|,t) + \gamma\left(\sup_{\tau\geq 0}\|\delta(\tau)\|\right)$$

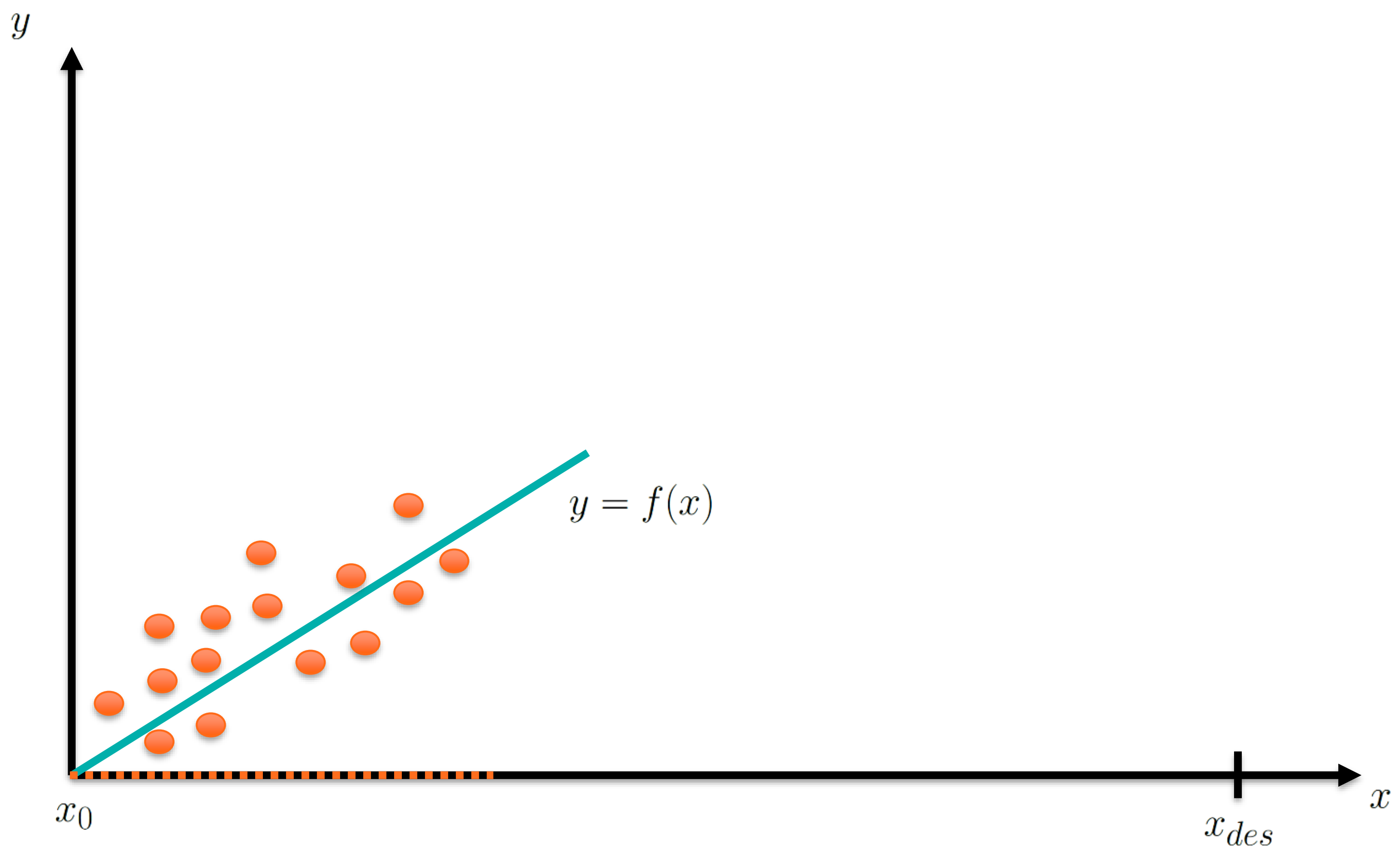
Taylor, Dorobantu, Le, Yue, Ames, A Control Lyapunov Perspective on Episodic Learning via Projection to State Stability, CDC 2019

Covariate Shift

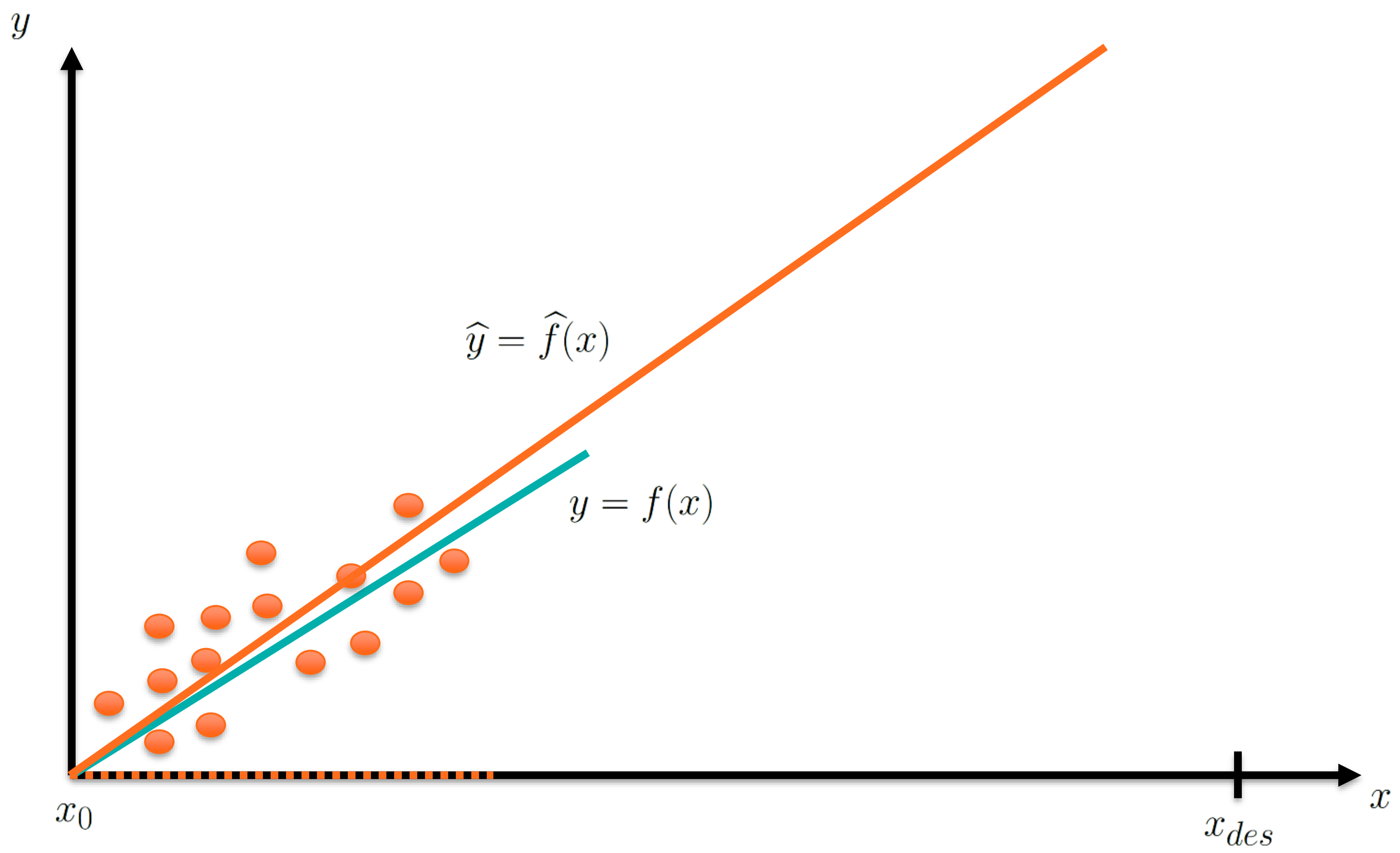
Covariate Shift



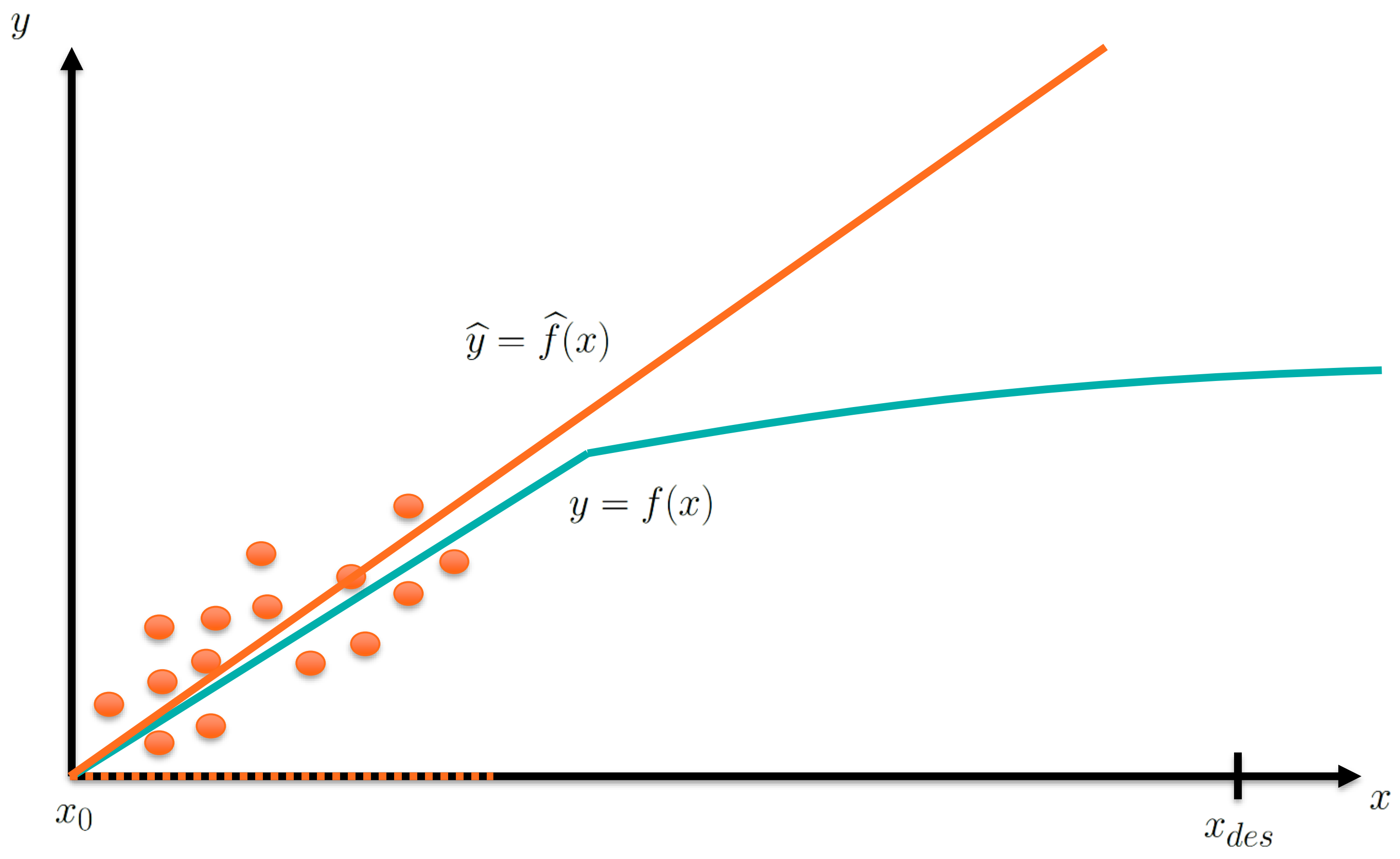
Covariate Shift



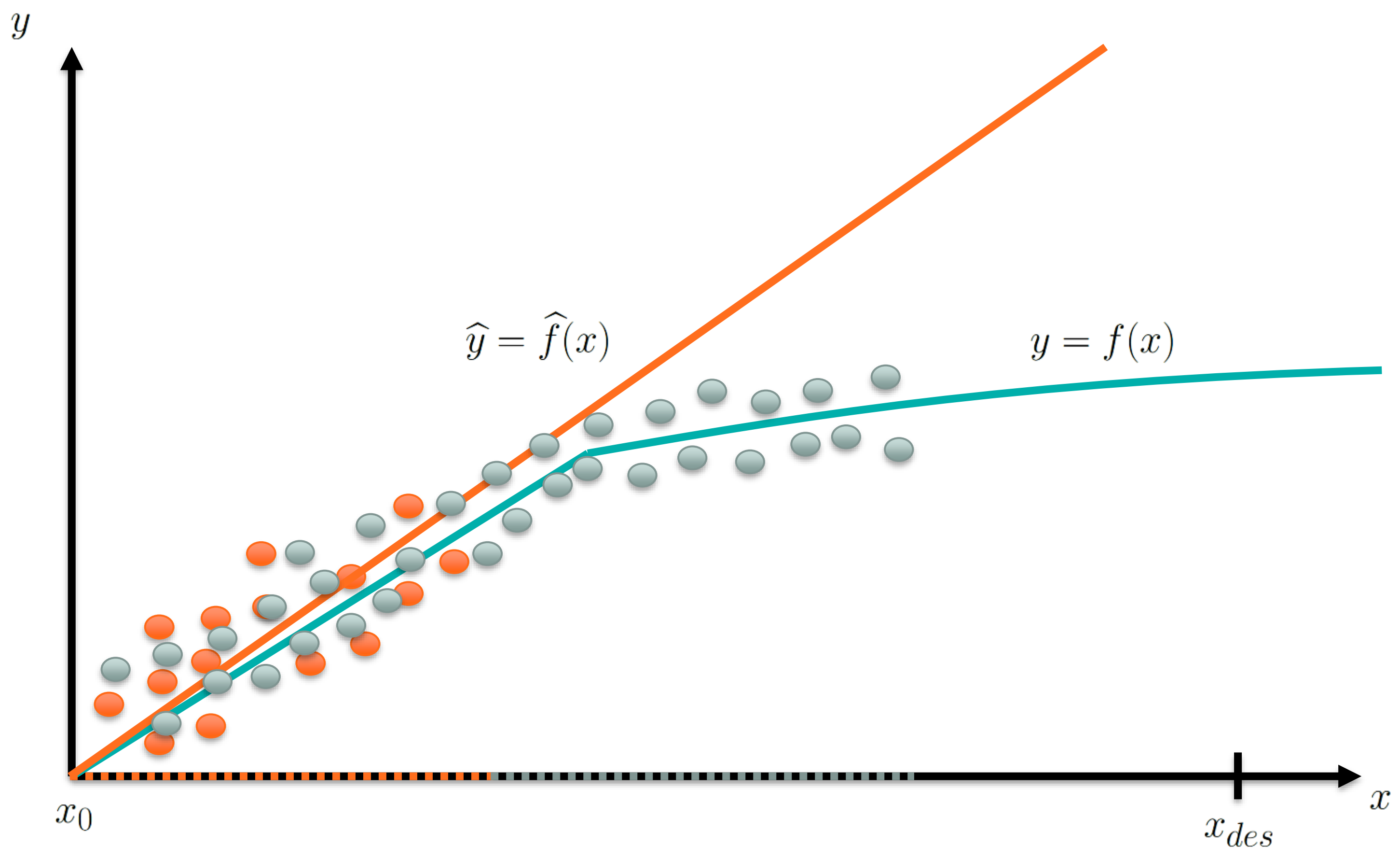
Covariate Shift



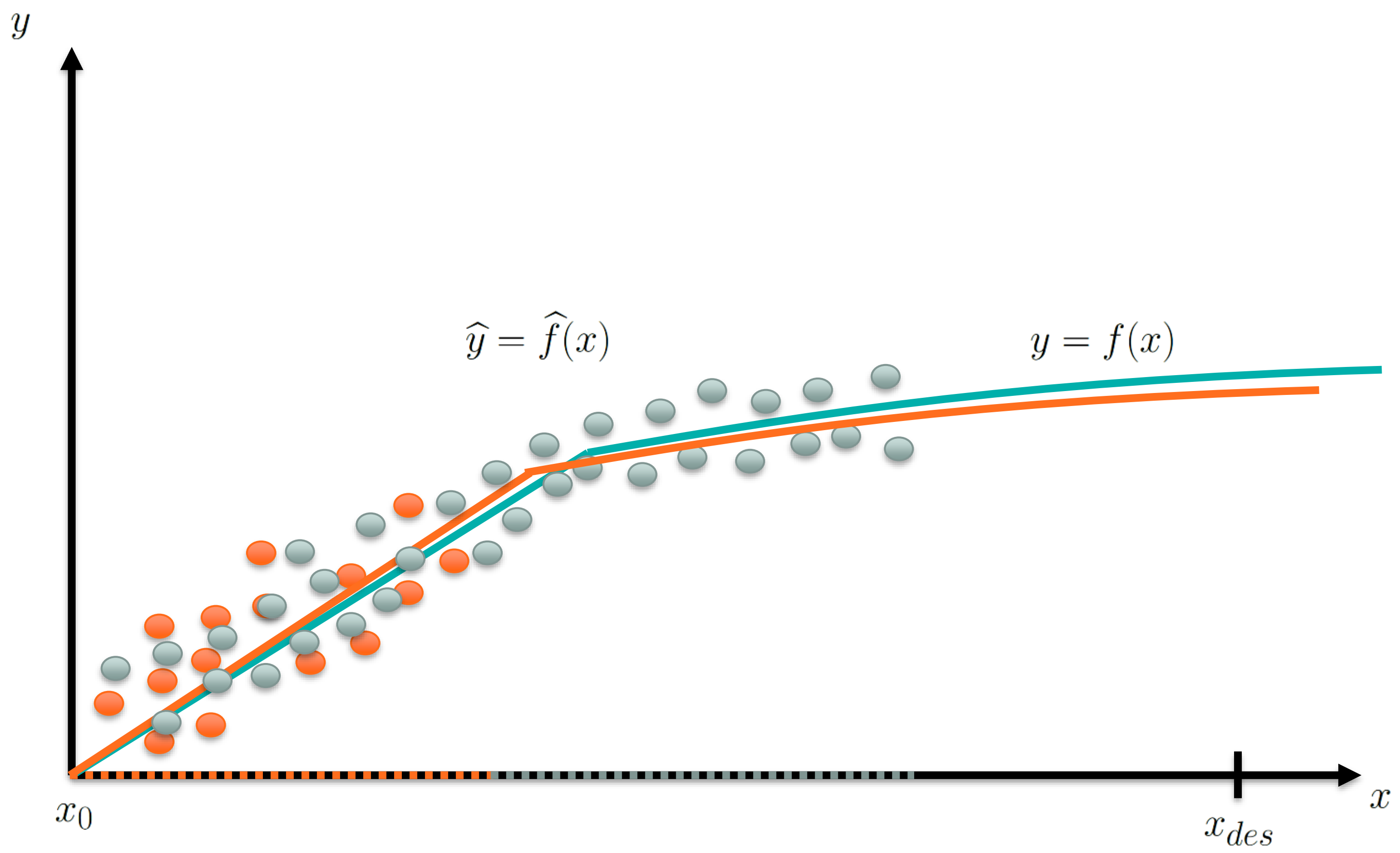
Covariate Shift



Covariate Shift



Covariate Shift



- Part I: Introducing the goal - “When using machine learning to reduce model uncertainty, what claims can be made on the stability of the resulting system?”
  - Introduce nonlinear affine dynamics, introduce Lyapunov and Control Lyapunov
  - Introduce uncertain nonlinear affine dynamics, show how they lead to model and residual Lyapunov dynamics.
  - Introduce the particular learning problem of learning CLF derivative. Show what the a_tilde and b_tilde terms appear once you have estimators, and indicate that this sets up our residual error analysis.
- Part II: Projection – to – State – Stability
  - Write down the definition of input-to-state stability, ISS-CLFs, forward invariance that comes with ISS-CLF
  - Highlighting our preceding learning structure, show that we really want a bound on in the state in terms of the norm of the disturbance in the CLF time derivative.
  - This motivates the construction of PSS. Describe what PSS is, state Theorem 1, Eq (10) and Eq (14)
  - State Corollary 1, connect back to Theorem 1 via quick walk proof sketch (get the implication right!!!)

- Part III: Uncertainty Function + PSS
- Part IV: Results
  - Consider an inverted pendulum system. Leave the majority of the details on what exactly the learning framework is to the paper.
  - Show that the inverted pendulum tracking performance becomes quite good compared to the PD controller
  - Show the heat maps that show mild improvement in the worst case bounds.
  - The conclusion here is very important. Essentially, there is a gap between good performance and certifying theoretical guarantees. We can get good performance without learning everything. But to make stronger claims on stability, we need more principled approaches for acquiring data. This analysis gives insight into what data holds value in acquiring when it comes to building these certificates.