Nonlinear Model Predictive Control of Robotic Systems with Control Lyapunov Functions

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Caltech
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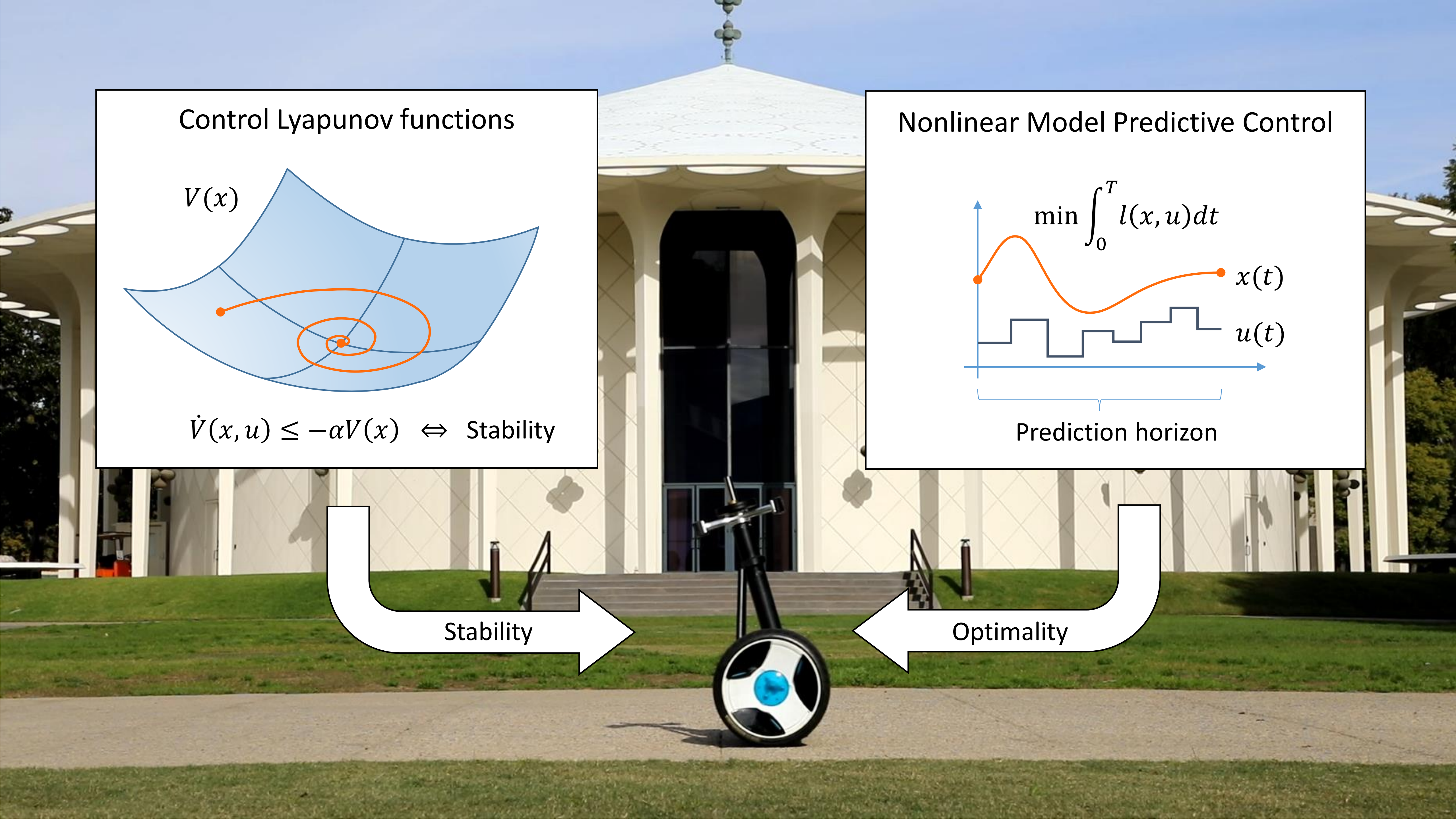
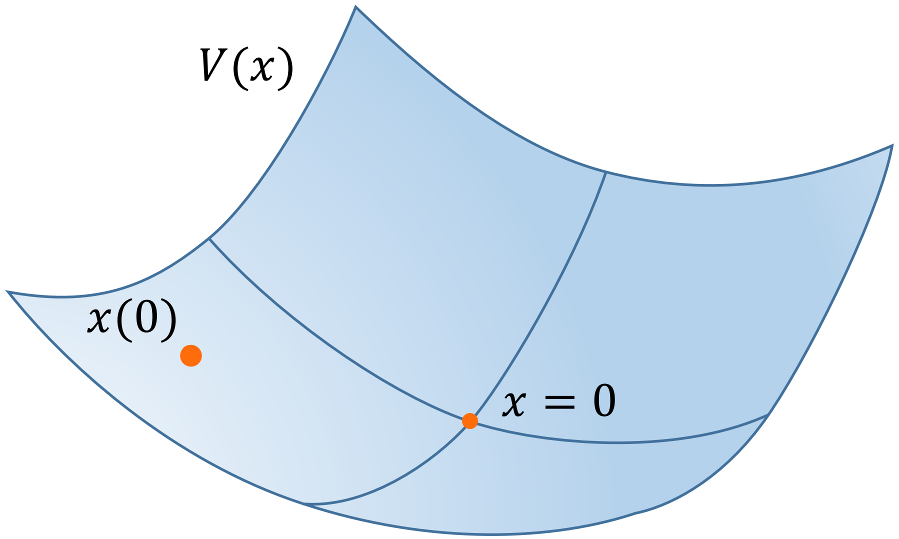

Control Lyapunov functions
$V(x)$
$\dot{V}(x,u) \leq -\alpha V(x) \iff$ Stability
Nonlinear Model Predictive Control
$\min \int_0^T l(x,u)dt$
$x(t)$
$u(t)$
Prediction horizon
Stability
Optimality

Control Lyapunov functions



- Dynamics: $x \in \mathbb{R}^n$ and u $\in \mathbb{R}^m$

$$\dot{x} = f(x) + g(x)u$$

- Lyapunov: $V : X \rightarrow \mathbb{R}_+$, satisfying:

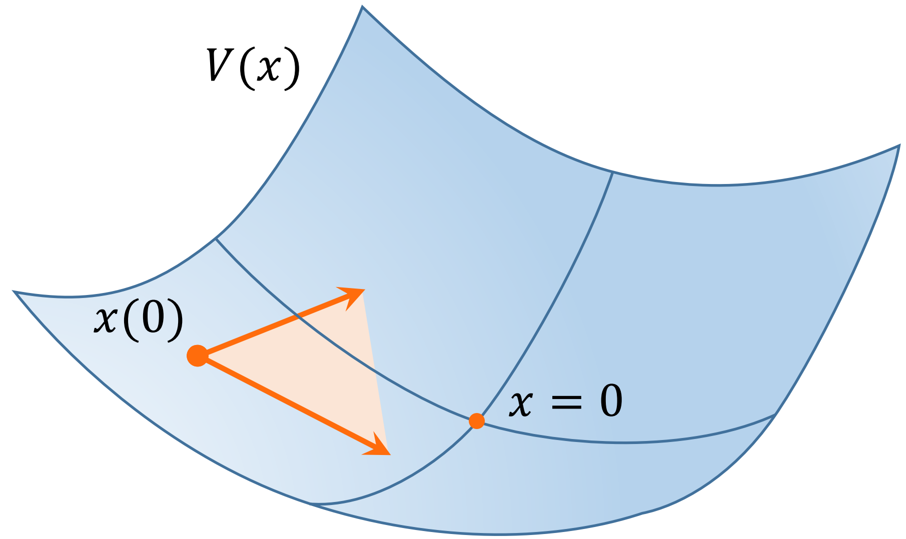
$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\inf_{u \in U} \dot{V}(x,u) \leq -\alpha V(x)$$

[1,2]

[1] Z. Artstein. Stabilization with relaxed controls. Nonlinear Analysis: Theory, Methods & Applications, 7(11): 1163–1173, 1983.
[2] R. A. Freeman and P. V. Kokotovic. Inverse optimality in robust stabilization. SIAM journal on control and optimization, 34(4):1365–1391, 1996.

Control Lyapunov functions



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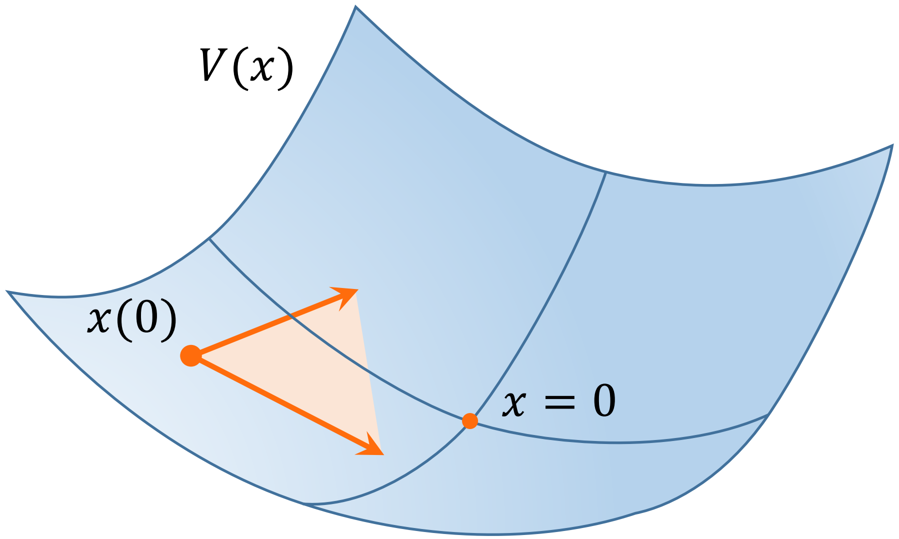
Control Lyapunov functions



[3]

CLF-QP:

$$u(x) = \text{argmin } \|u - u_{des}(x)\|^2$$

$$\text{s.t.} \quad \dot{V}(x,u) \leq -\alpha V(x)$$

- Dynamics: $x \in \mathbb{R}^n$ and u $\in \mathbb{R}^m$

$$\dot{x} = f(x) + g(x)u$$

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$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\inf_{u \in U} \dot{V}(x,u) \leq -\alpha V(x)$$

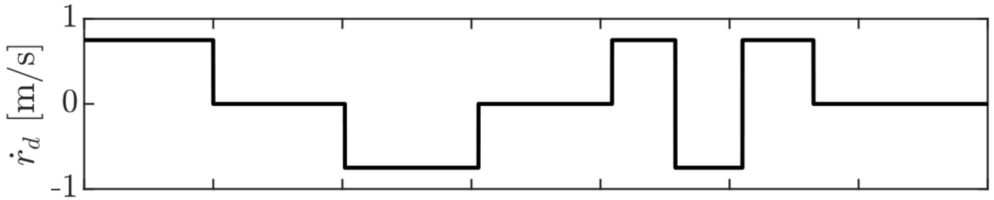
- Stabilizing set of controllers

$$u(x) \in \{u \in U \mid \dot{V}(x,u) \leq -\alpha V(x)\}$$
$$\Downarrow$$
$$V(x(t)) \leq e^{-\alpha t} V(x(0)) \implies x \rightarrow 0$$

[3] A. D. Ames and M. Powell. Towards the unification of locomotion and manipulation through control lyapunov functions and quadratic programs. 2013.

Baseline Experiments

Tracking linear velocity commands

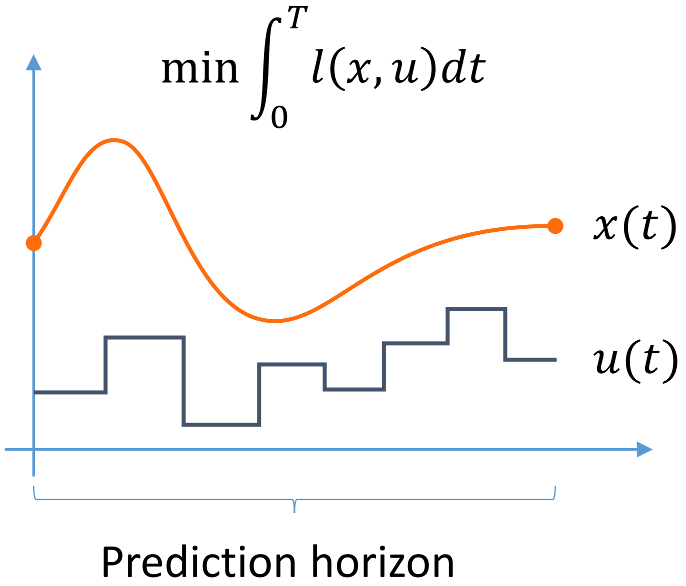


CLF-QP:

$$u(x) = \text{argmin } \|u\|^2$$

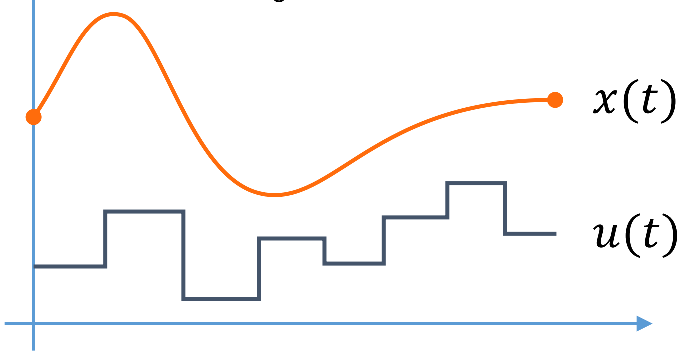
$$\text{s.t.} \quad \dot{V}(x,u) \leq -\alpha V(x)$$

Nonlinear Model Predictive Control



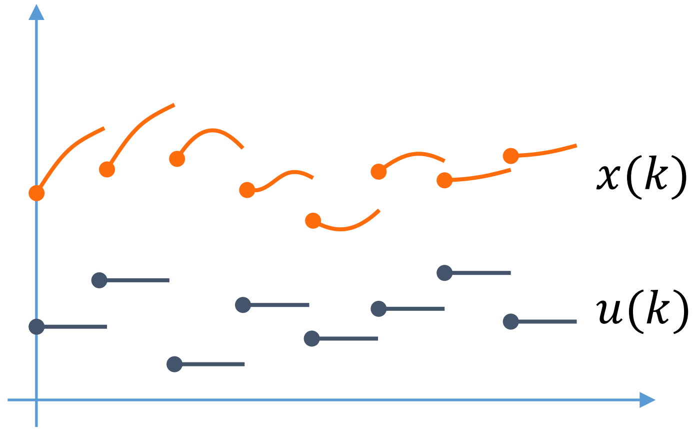
Nonlinear Model Predictive Control

$$\min \int_0^T l(x,u)dt$$



Discretization
(Direct multiple shooting) [4]

$$x_{k+1} = x_k + \int_{t_k}^{t_k+\delta t} f(x(\tau)) + g(x(\tau))u_k \,\mathrm{d}\tau$$



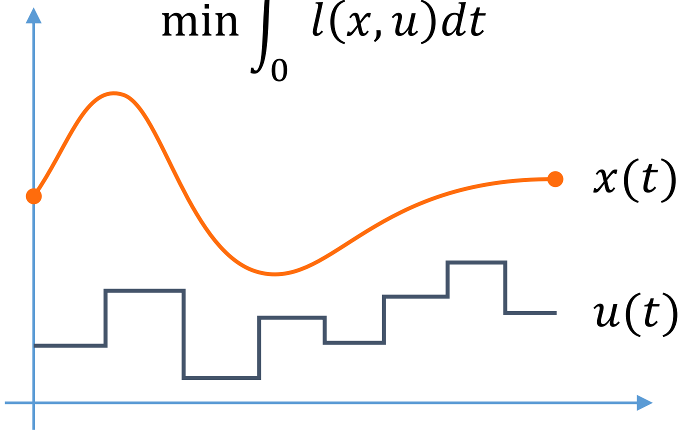
Nonlinear optimization problem

$$\begin{aligned}
\min_{X,U,S}\ & l_N(x_N,p) + \phi(s_N) + \sum_{k=0}^{N-1} l_k(x_k,u_k,p) + \phi(s_k) \\
\text{s.t.}\ & x_0 - \hat{x} = 0, \\
& x_{k+1} - f_k^d(x_k,u_k) = 0, \qquad k = 0,\dots,N-1, \\
& h_k(x_k,u_k,p) \le s_k, \qquad k = 0,\dots,N-1, \\
& h_N(x_N,p) \le s_N, \\
& s_k \ge 0, \qquad k = 0,\dots,N,
\end{aligned}$$

[4] H.G. Bock and K.J. Plitt. A Multiple Shooting Algorithm for Direct Solution of Optimal Control Problems. IFAC Proceedings Volumes, 17(2):1603 – 1608, 1984.

Nonlinear Model Predictive Control

$$\min \int_0^T l(x,u)dt$$



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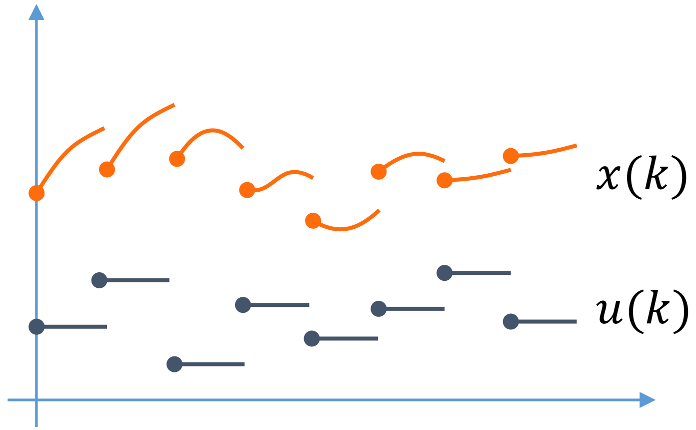

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& x_{k+1} - f_k^d(x_k,u_k) = 0, \quad k = 0,\dots,N-1, \\
& h_k(x_k,u_k,p) \leq s_k, \quad k = 0,\dots,N-1, \\
& h_N(x_N,p) \leq s_N, \\
& s_k \geq 0, \quad k = 0,\dots,N,
\end{aligned}$$

[5]

Solved with Sequential Quadratic Programming

- 1 iteration per control update
- Quadratic approximation of the Lagrangian
- Linear approximation of dynamics and constraints

[5] M. Diehl, et. al. Real-time optimization and nonlinear model predictive control of processes governed by differential-algebraic equations. Journal of Process Control, 2002.

Baseline Experiments

Tracking cost

Input cost

NMPC-β:

$$
\begin{aligned}
\min_{X,U} \quad & \beta V(x_N, t_N) + \sum_{k=0}^{N-1} \eta(x_k, t_k)^\top Q \eta(x_k, t_k) + \frac{1}{2} u_k^\top u_k \\
\text{s.t} \quad & x_0 - \hat{x} = 0, \\
& x_{k+1} - f_k^d(x_k, u_k) = 0, \qquad k = 0, \dots, N-1, \\
& \underline{u} \le u_k \le \bar{u}, \qquad k = 0, \dots, N-1,
\end{aligned}
$$

Tuning weight on terminal tracking cost

CLF-NMPC :

Constraining $\dot{V}(x,u)$

Input cost

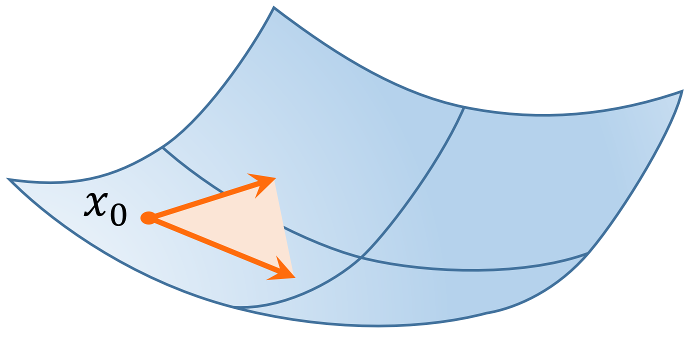
(slack penalty)

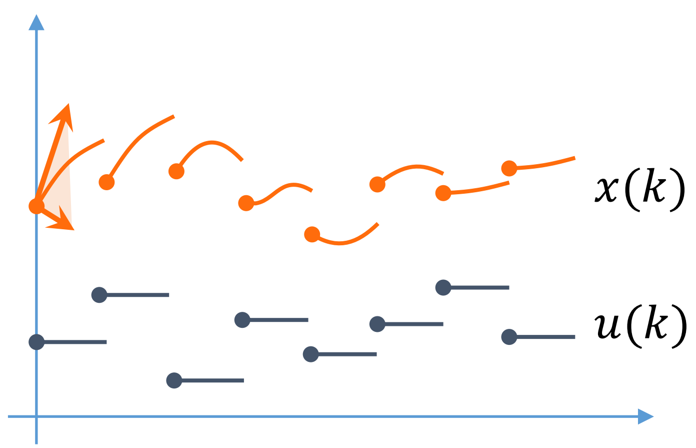
$$\begin{aligned}
\min_{X,U,S} \quad & \phi(s_N) + \sum_{k=0}^{N-1} \frac{1}{2} u_k^\top u_k + \phi(s_k) \\
\text{s.t} \quad & x_0 - \hat{x} = 0, \\
& x_{k+1} - f_k^d(x_k, u_k) = 0, \qquad k = 0, \dots, N-1, \\
& \underline{u} \le u_k \le \bar{u}, \qquad k = 0, \dots, N-1, \\
& s_k \ge 0, \qquad k = 0, \dots, N, \\
\textit{CLF-0}: \quad & h_{CLF}(\hat{x}, u_0) \le s_0,
\end{aligned}$$

CLF stability constraint

$h_{CLF}(x,u)$:

$\dot{V}(x,u) \le -\alpha V(x)$





CLF-NMPC :

Constraining $\dot{V}(x,u)$

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\min_{X,U,S} \quad & \phi(s_N) + \sum_{k=0}^{N-1} \frac{1}{2} u_k^\top u_k + \phi(s_k) \\
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& x_{k+1} - f_k^d(x_k, u_k) = 0, \qquad k = 0, \dots, N-1, \\
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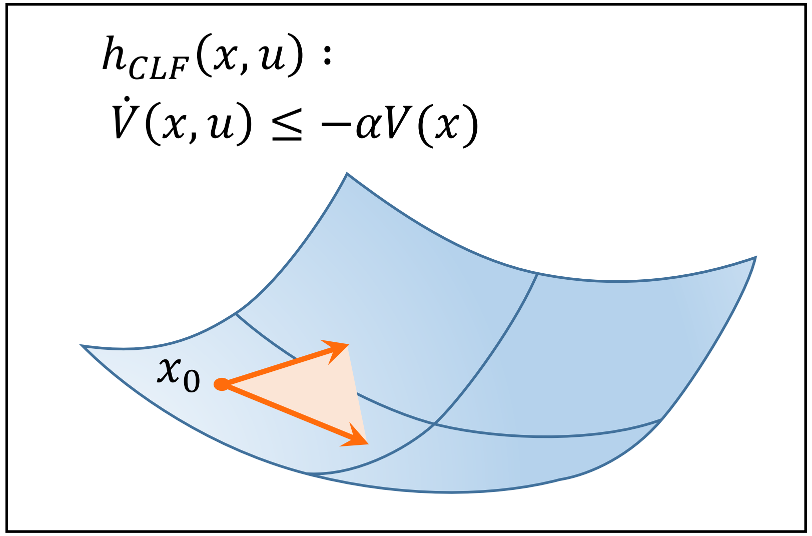
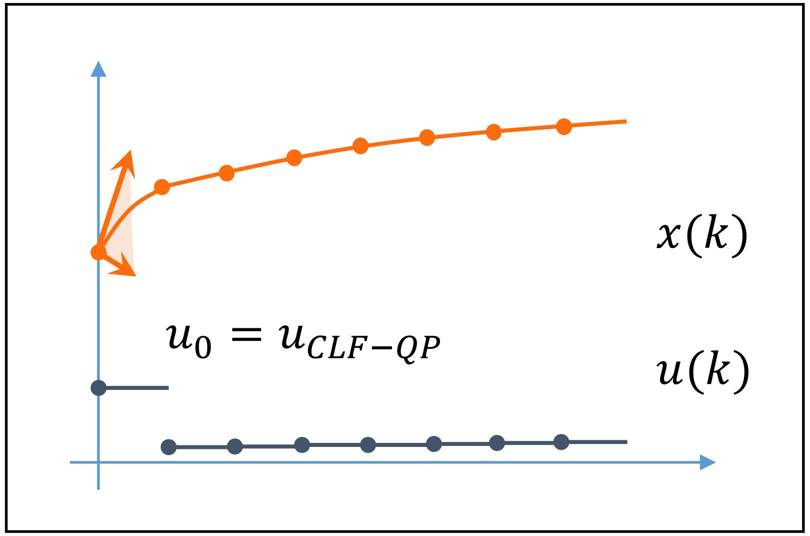
CLF stability constraint



Stabilizing, but no performance gain



CLF-NMPC :

Constraining $\dot{V}(x,u)$

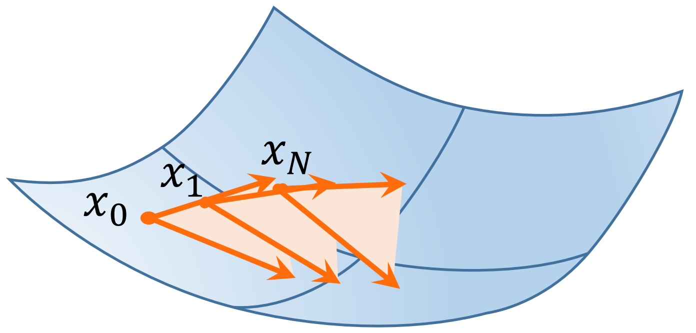
$$
\begin{aligned}
\min_{X,U,S} \quad & \phi(s_N) + \sum_{k=0}^{N-1} \frac{1}{2} u_k^\top u_k + \phi(s_k) \\
\text{s.t} \quad & x_0 - \hat{x} = 0, \\
& x_{k+1} - f_k^d(x_k, u_k) = 0, \qquad k = 0, \dots, N-1, \\
& \underline{u} \leq u_k \leq \bar{u}, \qquad k = 0, \dots, N-1, \\
& s_k \geq 0, \qquad k = 0, \dots, N, \\
\textit{CLF-0}: \quad & h_{CLF}(\hat{x}, u_0) \leq s_0,
\end{aligned}
$$

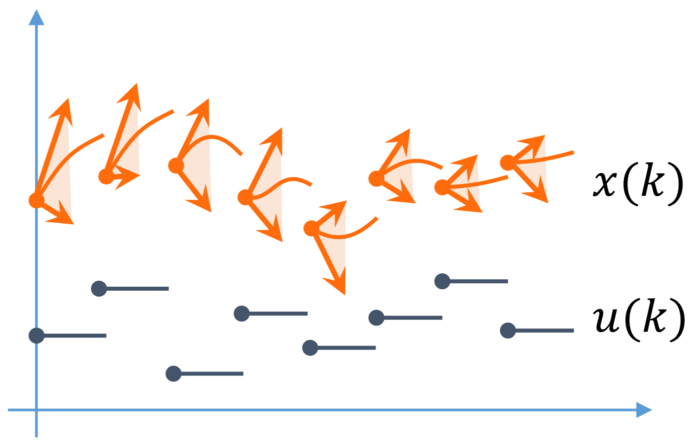
Additionally, for

$$\textit{CLF-ALL}: \quad h_{CLF}(x_k, u_k) \leq s_k, \qquad k = 1, \dots, N-1,$$

$h_{CLF}(x,u)$:

$\dot{V}(x,u) \leq -\alpha V(x)$

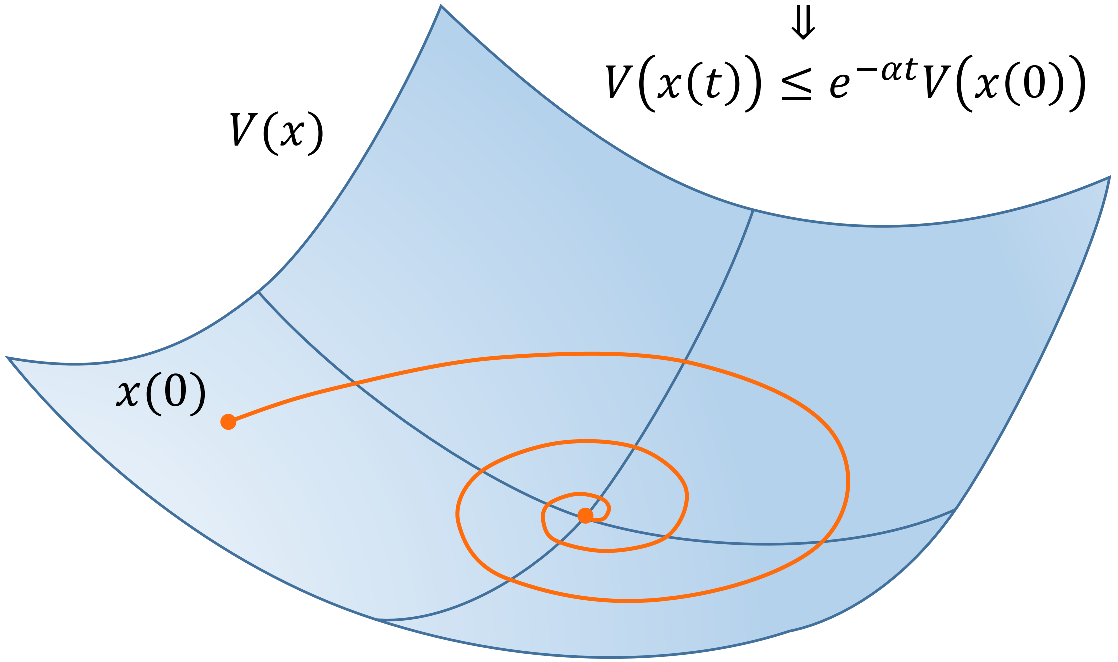




CLF-NMPC :

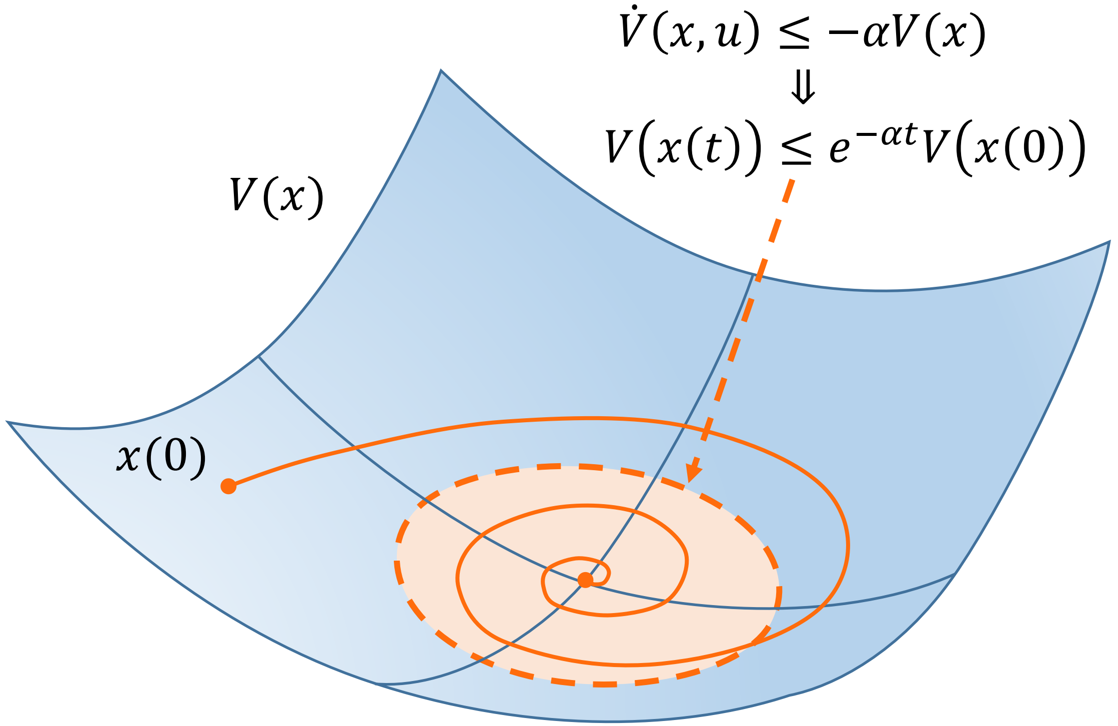
Level set constraints

$$\dot{V}(x,u) \leq -\alpha V(x)$$
$$\Downarrow$$
$$V\big(x(t)\big) \leq e^{-\alpha t} V\big(x(0)\big)$$



CLF-NMPC :

Level set constraints



CLF-NMPC :

Level set constraints

$$
\begin{aligned}
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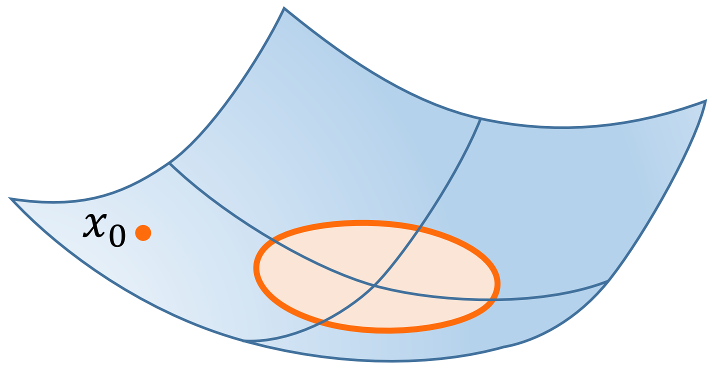
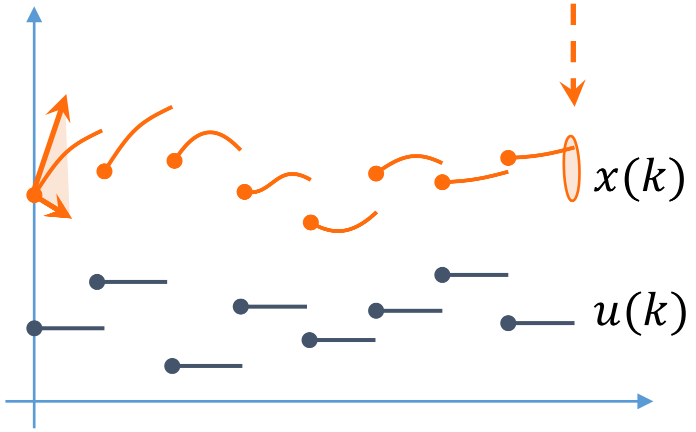
& x_{k+1} - f_k^d(x_k, u_k) = 0, \qquad k = 0, \dots, N-1, \\
& \underline{u} \leq u_k \leq \bar{u}, \qquad k = 0, \dots, N-1, \\
& s_k \geq 0, \qquad k = 0, \dots, N, \\
\textit{CLF-0}: \quad & h_{CLF}(\hat{x}, u_0) \leq s_0,
\end{aligned}
$$

Additionally, for

$$\textit{LLS-N}: \quad h_{LLS}(x_N, \hat{x}) \leq s_N,$$

$h_{LLS}(x): \quad V\big(x(t)\big) \leq e^{-\alpha t} V\big(x(0)\big)$





CLF-NMPC :

Level set constraints

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\min_{X,U,S} \quad & \phi(s_N) + \sum_{k=0}^{N-1} \frac{1}{2} u_k^\top u_k + \phi(s_k) \\
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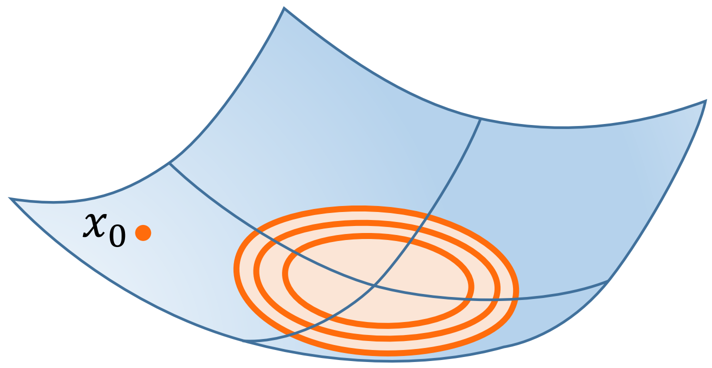
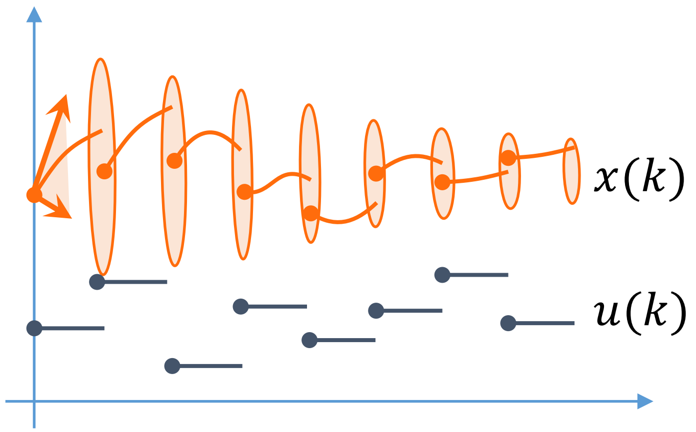
& x_{k+1} - f_k^d(x_k, u_k) = 0, \qquad k = 0, \dots, N-1, \\
& \underline{u} \leq u_k \leq \bar{u}, \qquad k = 0, \dots, N-1, \\
& s_k \geq 0, \qquad k = 0, \dots, N, \\
\textit{CLF-0}: \quad & h_{CLF}(\hat{x}, u_0) \leq s_0,
\end{aligned}
$$

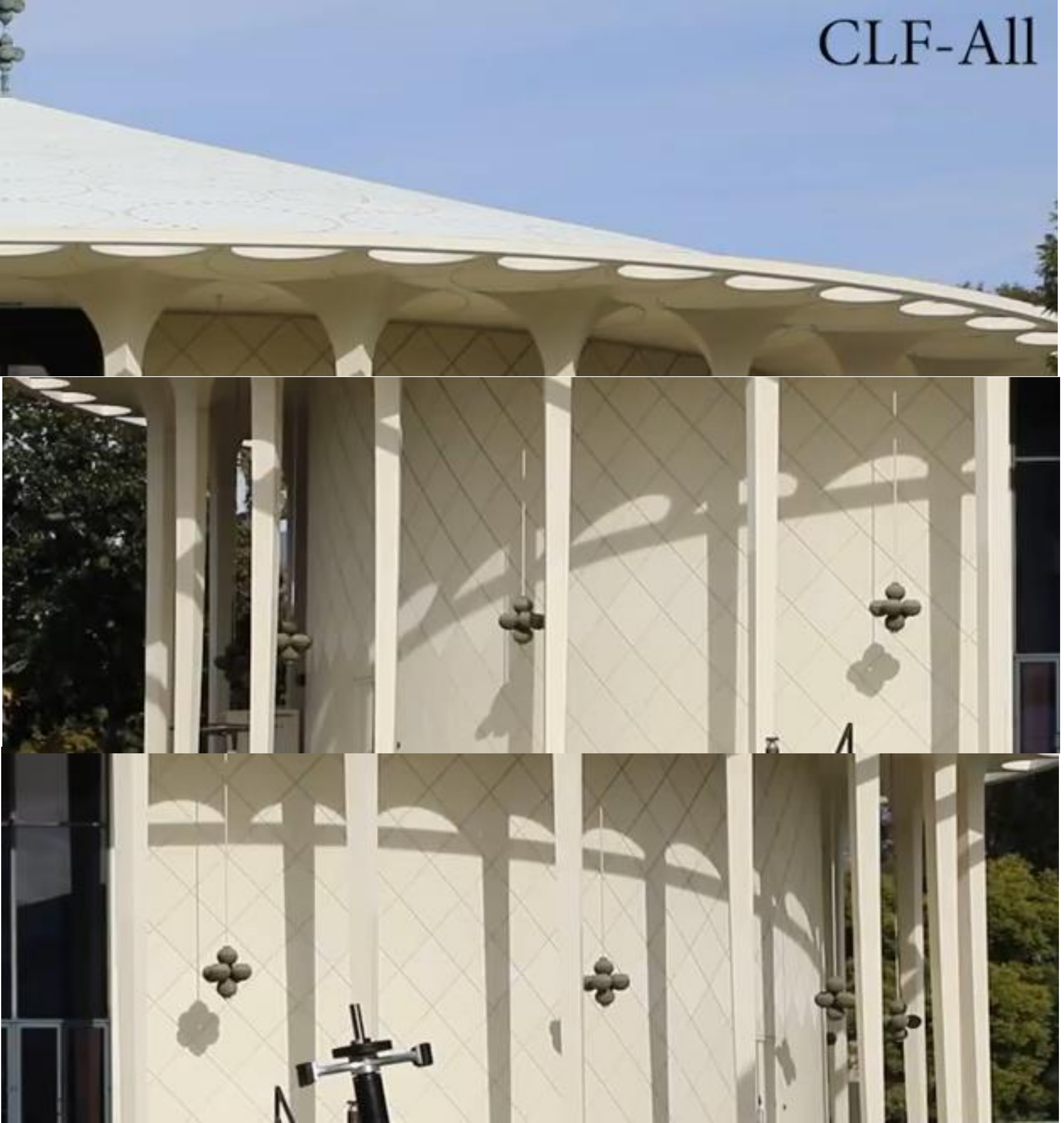
Additionally, for

$$
\textit{LLS-All}: \quad h_{LLS}(x_k, \hat{x}) \leq s_k, \qquad k = 1, \dots, N.
$$

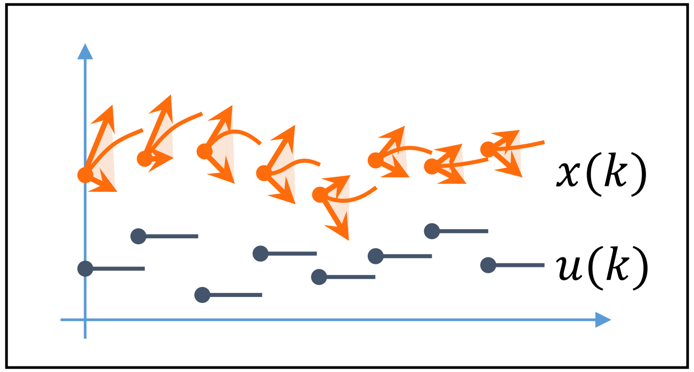
$h_{LLS}(x): \quad V(x(t)) \leq e^{-\alpha t} V(x(0))$



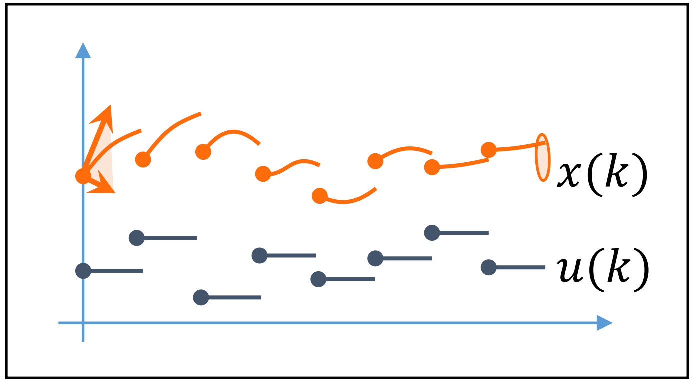




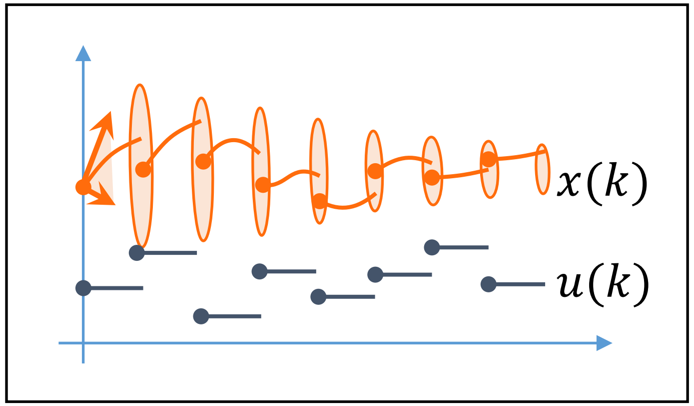
$$u_{avg} = 1.594\ [A]$$
$$t_{CPU} = 5.56\ [ms]$$



$$u_{avg} = 1.666\ [A]$$
$$t_{CPU} = 4.17\ [ms]$$



$$u_{avg} = 1.898\ [A]$$
$$t_{CPU} = 6.13\ [ms]$$




CLF-All

Simulation ($u_{avg}[A]$)

N	1	10	20	30	40	50
CLF-QP	1.085	1.085	1.085	1.085	1.085	1.085
CLF-0	1.085	1.085	1.085	1.085	1.085	1.085
CLF-All	1.085	1.072	0.952	0.849	**0.794**	**0.769**
LLS-N	1.083	0.957	0.889	0.842	0.808	0.784
LLS-All	1.083	0.956	0.887	0.839	0.805	0.782
NMPC-0.1	-	-	3.232	2.435	2.036	1.783
NMPC-1	-	3.026	2.019	1.732	1.574	1.471
NMPC-10	**0.828**	**0.607**	**0.704**	**0.823**	0.926	1.006

Main takeaways

- The CLF-NMPC controllers are stable for any horizon length (N).
- Outperform the CLF-QP formulation.
- For the baseline NMPC stability and performance are coupled and depend on tuning.
- First time for a combined approach to be demonstrated on hardware.


CLF-All
Caltech
ETH zürich