Safe Controller Synthesis with Tunable Input-to-State Safe Control Barrier Functions

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Background

- Consider the system

$$\dot{x} = f(x) + g(x)u$$

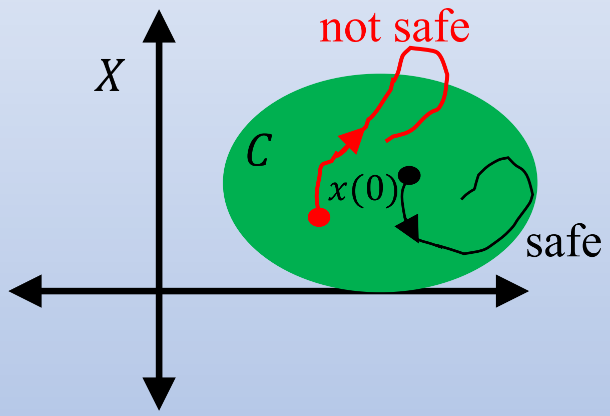
where state $x \in X \subseteq R^n$ and input $u \in R^m$

Define a *safe set* in X

$$C = \{x \in X \mid h(x) \geq 0\}$$

where $h: R^n \to R$ is continuously differentiable.

Safety: If for any initial condition $x(0) \in C$, $x(t) \in C$ for all t (i.e., C is forward invariant), then system is *safe* with respect to C.



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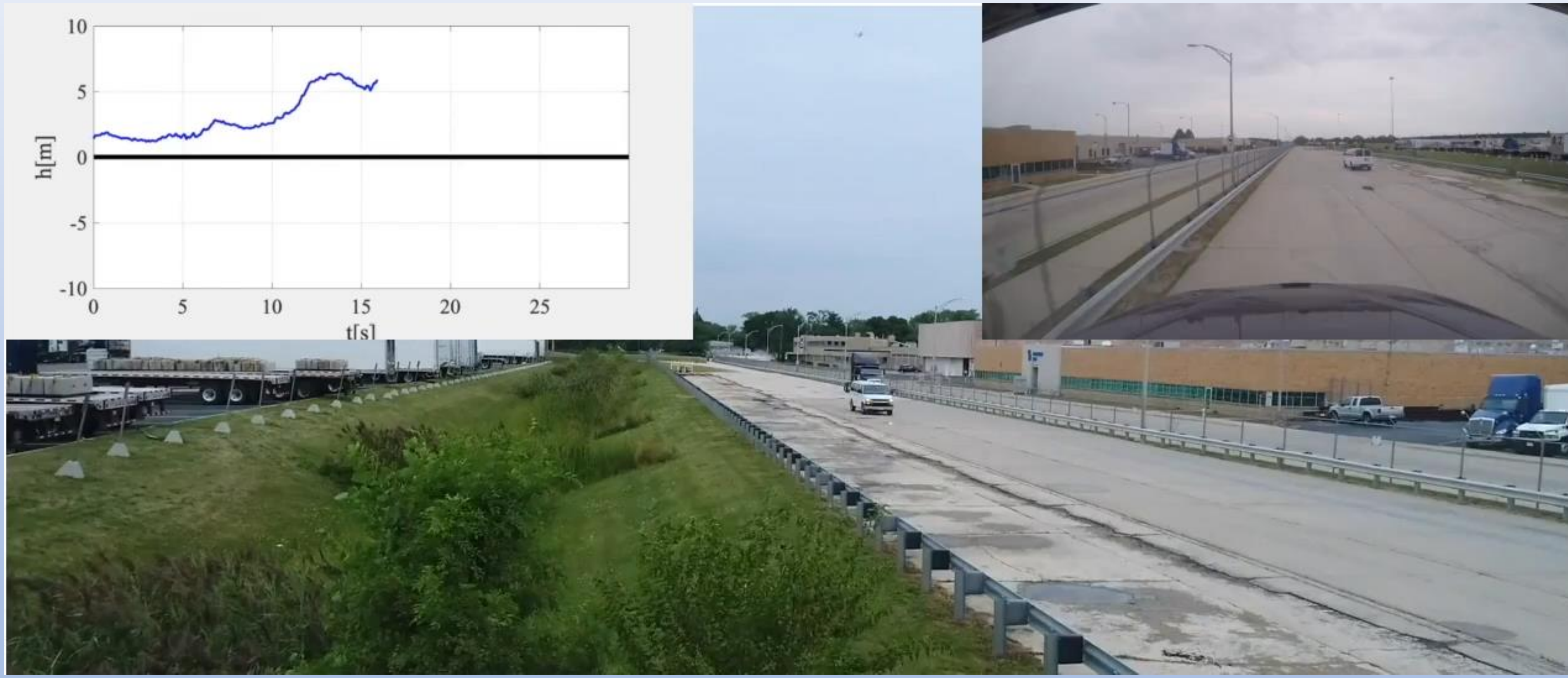


Background

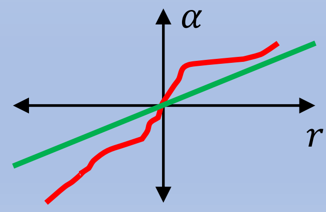
<u>Control Barrier Function (CBF)</u>: The function h is a *CBF* if there exists a $\alpha \in K_\infty^{\mathrm{e}}$ such that this satisfies:

$$\sup_{u \in R^m} \dot{h}(x,u) = \sup_{u \in R^m}\left[\frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x) u\right] > -\alpha\big(h(x)\big)$$

Extended class K_∞ (K_∞^{e})

$\alpha: R \to R$

- Continuous
- $\alpha(0) = 0$
- Monotonically increasing
- $\lim_{r \to \infty} \alpha(r) \to \infty$ and
- $\lim_{r \to -\infty} \alpha(r) \to -\infty$



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Control Barrier Function (CBF): The function h is a *CBF* if there exists a $\alpha_c > 0$ such that this satisfies:

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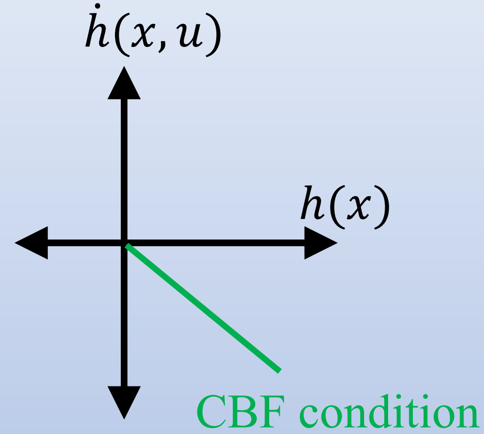

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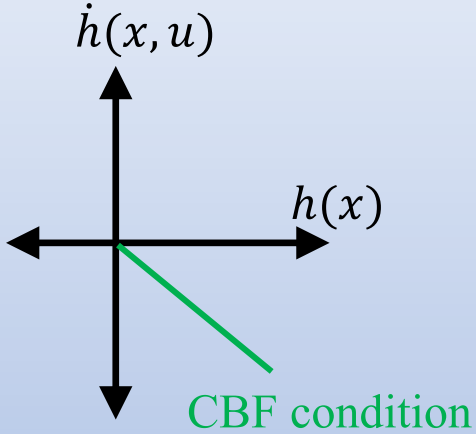

We define the point-wise set of controllers:

$$K_{\mathrm{cbf}}(x) = \left\{ u \in R^m \,\middle|\, \dot{h}(x,u) \geq -\alpha_c h(x) \right\}$$

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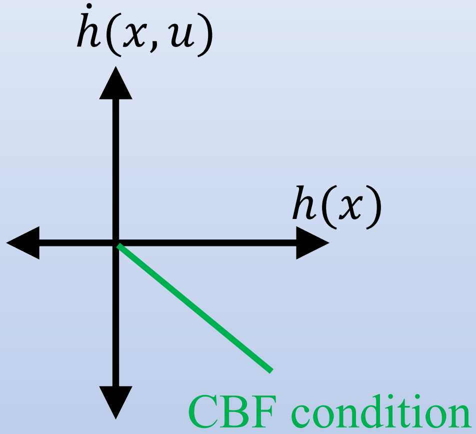

We define the point-wise set of controllers:

$$K_{\text{cbf}}(x) = \{ u \in R^m \mid \dot{h}(x,u) \geq -\alpha_c h(x) \}$$

<u>Theorem[1]</u>: Any controller $k(x) \in K_{\text{cbf}}(x)$ renders system safe.

[1] Ames et. al, “Control Barrier Function Based Quadratic Programs for Safety Critical Systems”, *IEEE Transactions On Automatic Control*, Vol. 62, No. 8, August 2017

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Example

Consider a double integrator system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = u$$

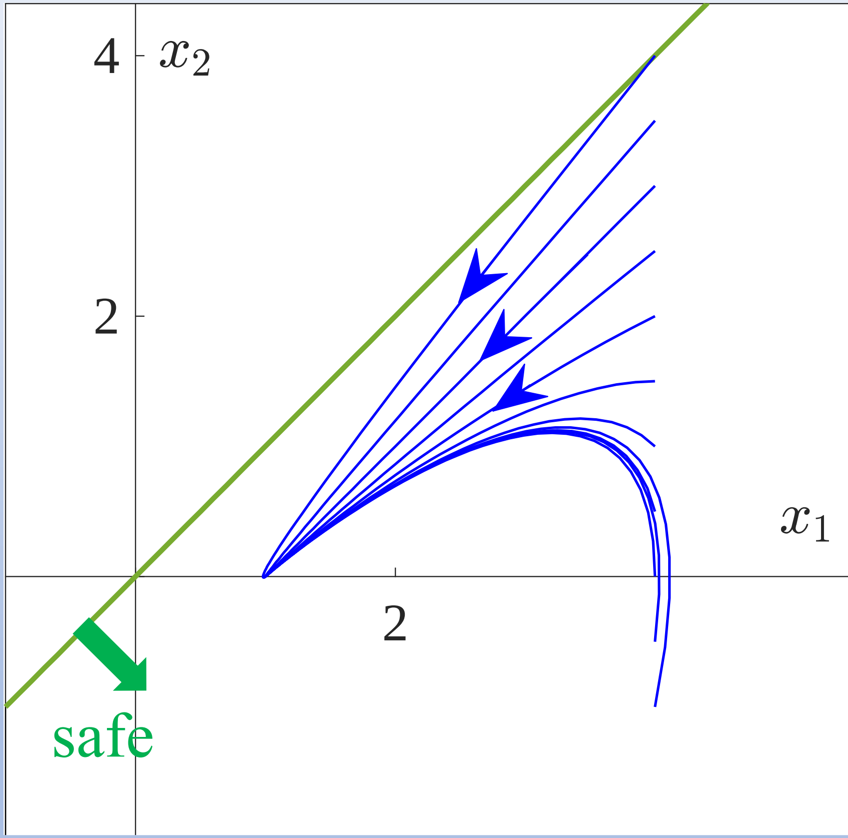
and safe set

$$C = \{x \in R^2 \mid \underbrace{x_1 - x_2}_{h(x)} \geq 0\}$$

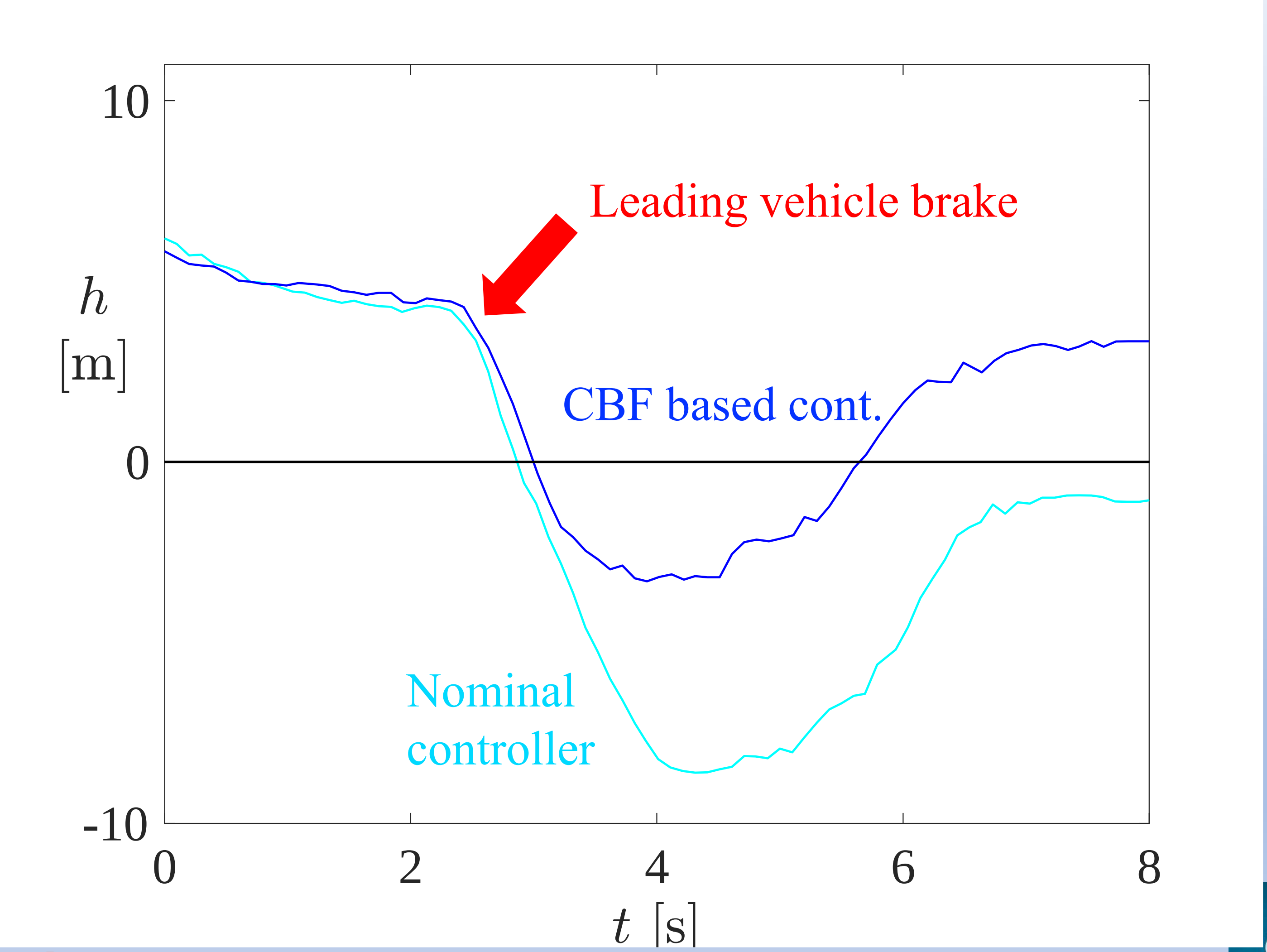
It can be shown the controller

$$k(x) = x_1 - 2x_2 - 1$$

renders system safe since $k(x) \in K_{\text{cbf}}(x)$ with $\alpha_c = 1$.



Background - Truck



Example

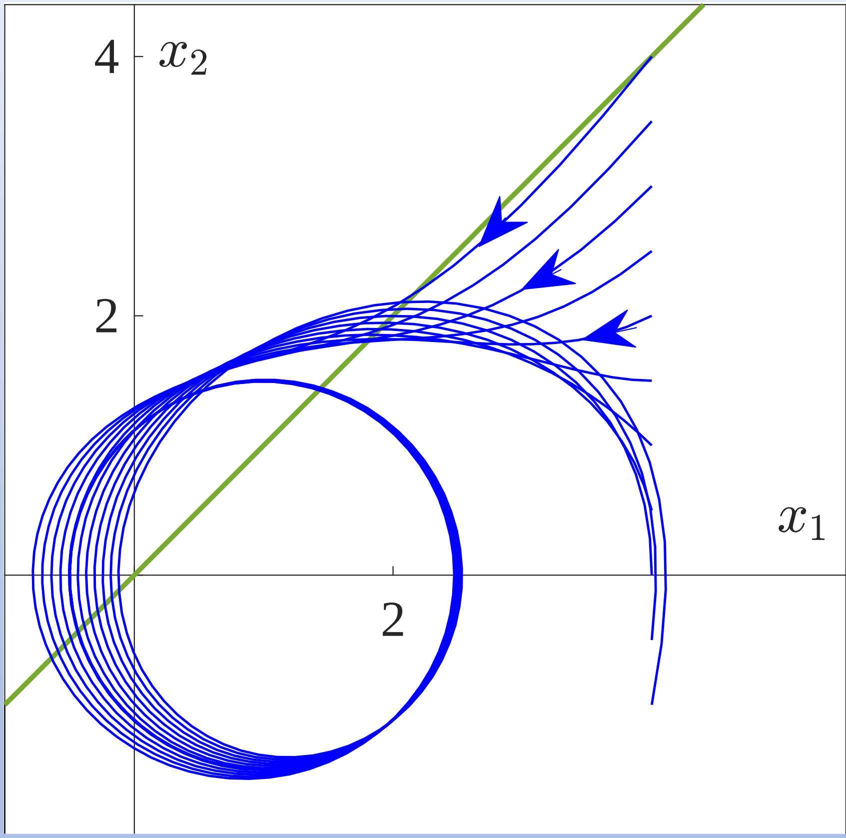
Consider a double integrator system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = u + d$$

where $\sup_{t \geq 0} \|d(t)\| = \|d\|_\infty \leq \delta.$

Consider the disturbance

$$d(t) = \delta \sin t$$



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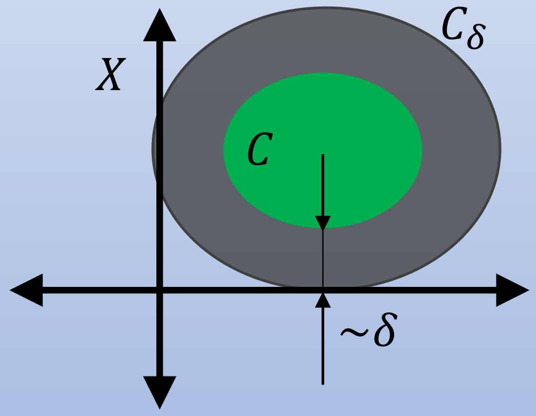
Input to State Safety

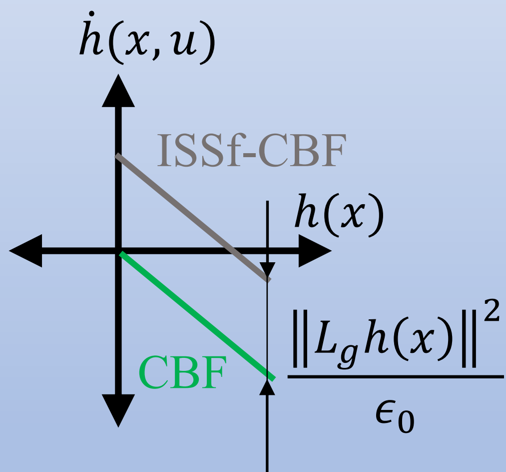
<u>Input-to-State Safety (ISSf)</u>: If there exists a larger set C_δ that is forward invariant in the presence of the disturbance, then the system is *input-to-state safe*. The set C is ISSf set.

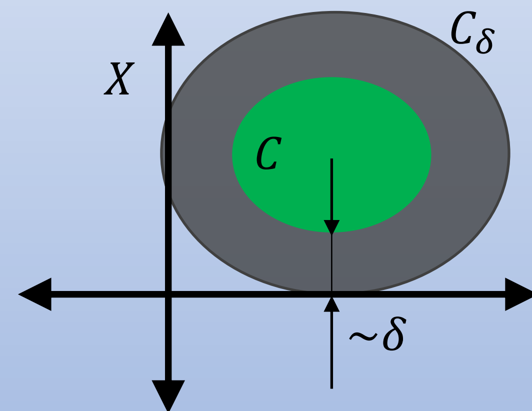


ISSf-CBF[2]

ISSf CBF:

$$\sup_{u \in R^m} \dot{h}(x,u) > -\alpha_c h(x) + \frac{\|L_g h(x)\|^2}{\epsilon_0}$$





[2] Kolathaya et. al, “Input-to-state safety with control barrier functions”, *IEEE Control Systems Letters*, Vol 3, pp 108–113, 2018.

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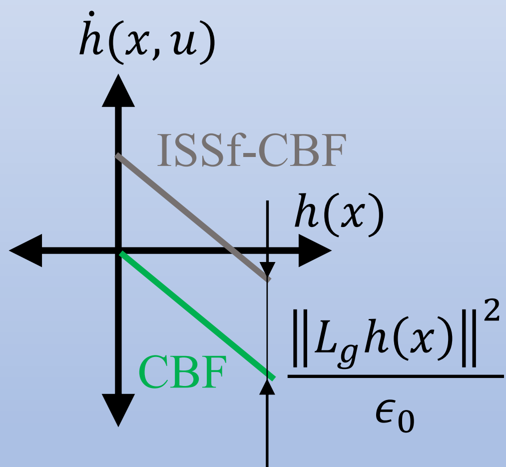
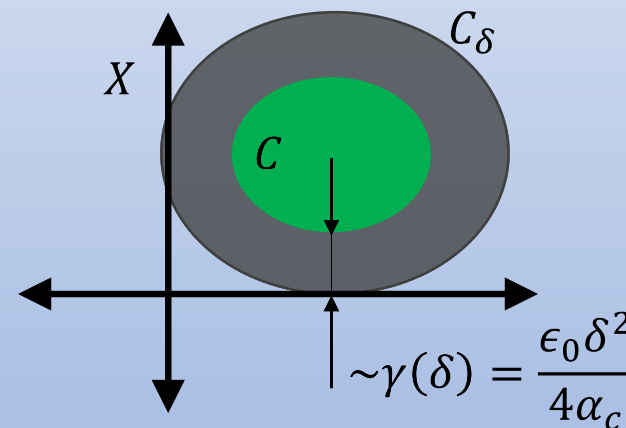


Theorem[2]: Controller $k(x)$ that satisfies ISSf-CBF condition ensures the set C_δ is forward invariant (i.e., C is *ISSf*)

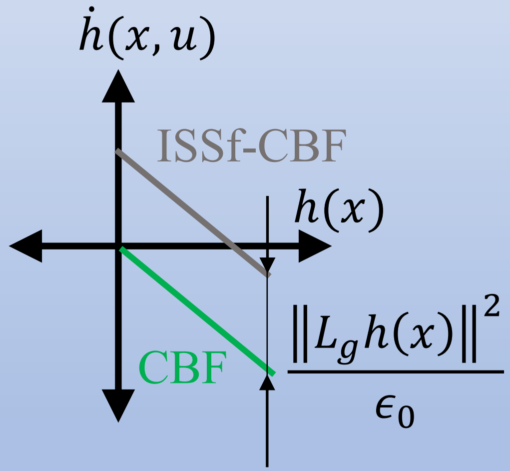
$$C_\delta = \{x \in X \mid h(x) + \gamma(\delta) \geq 0\}$$

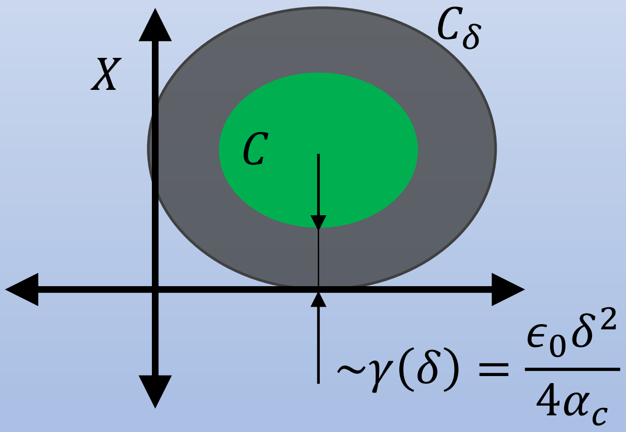


[2] Kolathaya et. al, “Input-to-state safety with control barrier functions”, *IEEE Control Systems Letters*, Vol 3, pp 108–113, 2018.

Motivation

<u>Trade-off:</u>
As $\epsilon_0 \uparrow$ → Less conservative
Large expansion



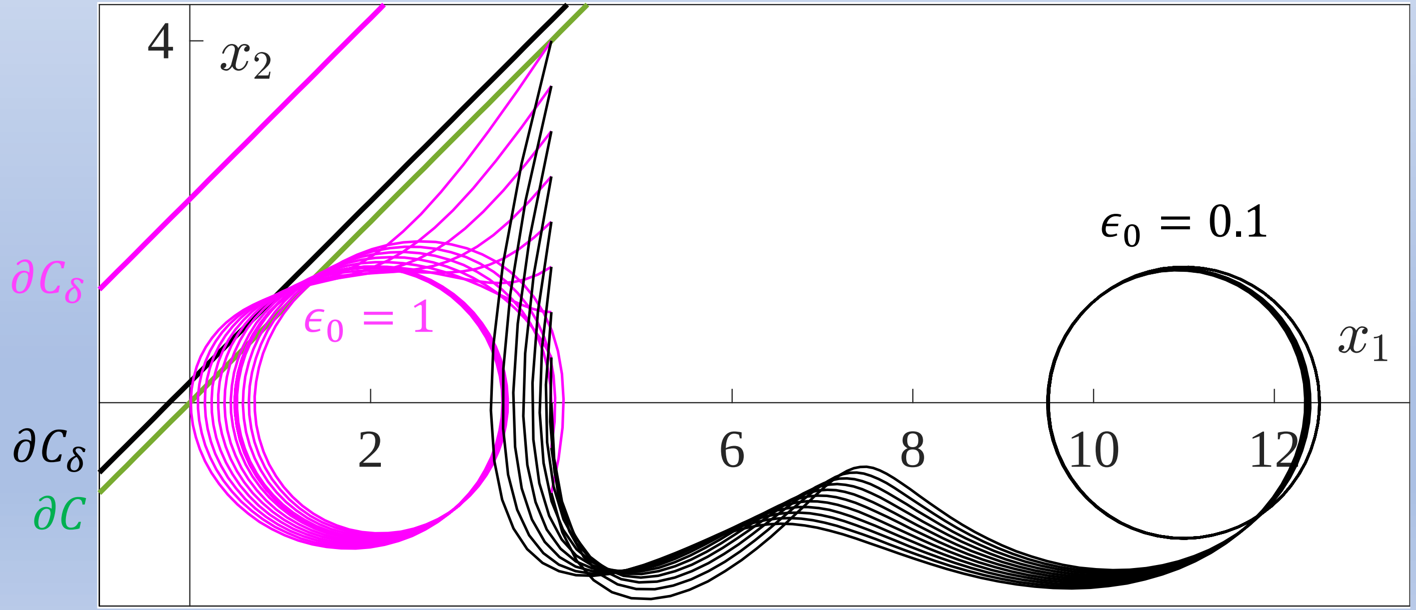


Motivation

Controller:

$$k_{\text{issf}}(x) = k(x) - \frac{1}{\epsilon_0}$$

can be shown to satisfy ISSf-CBF condition.



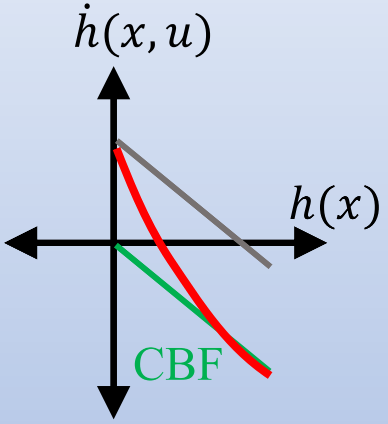
Tunable ISSf-CBF

The CBF h gives a *measure of safety*.

On the safe set boundary → $h(x) = 0$

Not very safety critical → $h(x) \gg 0$

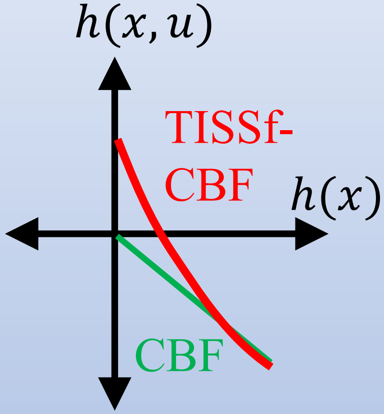
Proposition: Parameterize ϵ with $h(x)$



Tunable ISSf-CBF

<u>TISSf-CBF</u>: The function h is an *TISSf-CBF* if there exists a $\alpha_c > 0$ and $\epsilon: R \to R_{>0}$ such that condition below satisfies

$$\sup_{u \in R^m} \dot{h}(x,u) > -\alpha_c h(x) + \frac{\|L_g h(x)\|^2}{\epsilon(h(x))}$$



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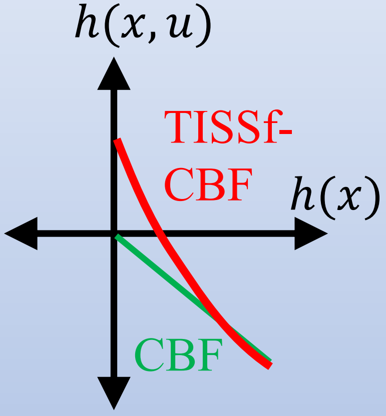

We define the point-wise set of controllers:

$$K_{\text{tissf}}(x) = \left\{ u \in R^m \,\middle|\, \dot{h}(x,u) \geq -\alpha_c h(x) + \frac{\|L_g h(x)\|^2}{\epsilon(h(x))} \right\}$$

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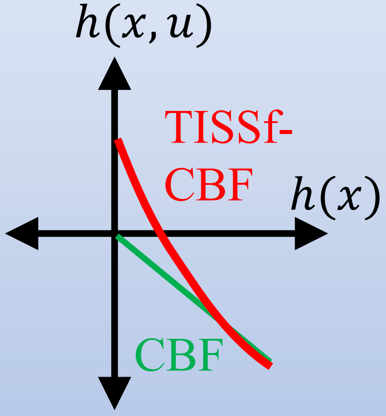

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<u>**Theorem**</u>**:** Any controller $k(x) \in K_{\mathrm{tissf}}(x)$ renders the set C_δ forward invariant (hence C ISSf set) if ${}^{\mathrm{d}\epsilon}/_{\mathrm{d}h} \geq 0$. Furthermore

$$\gamma_{\mathrm{T}}(h(x), \delta) = \frac{\epsilon(h(x))\delta^2}{4\alpha_c}$$

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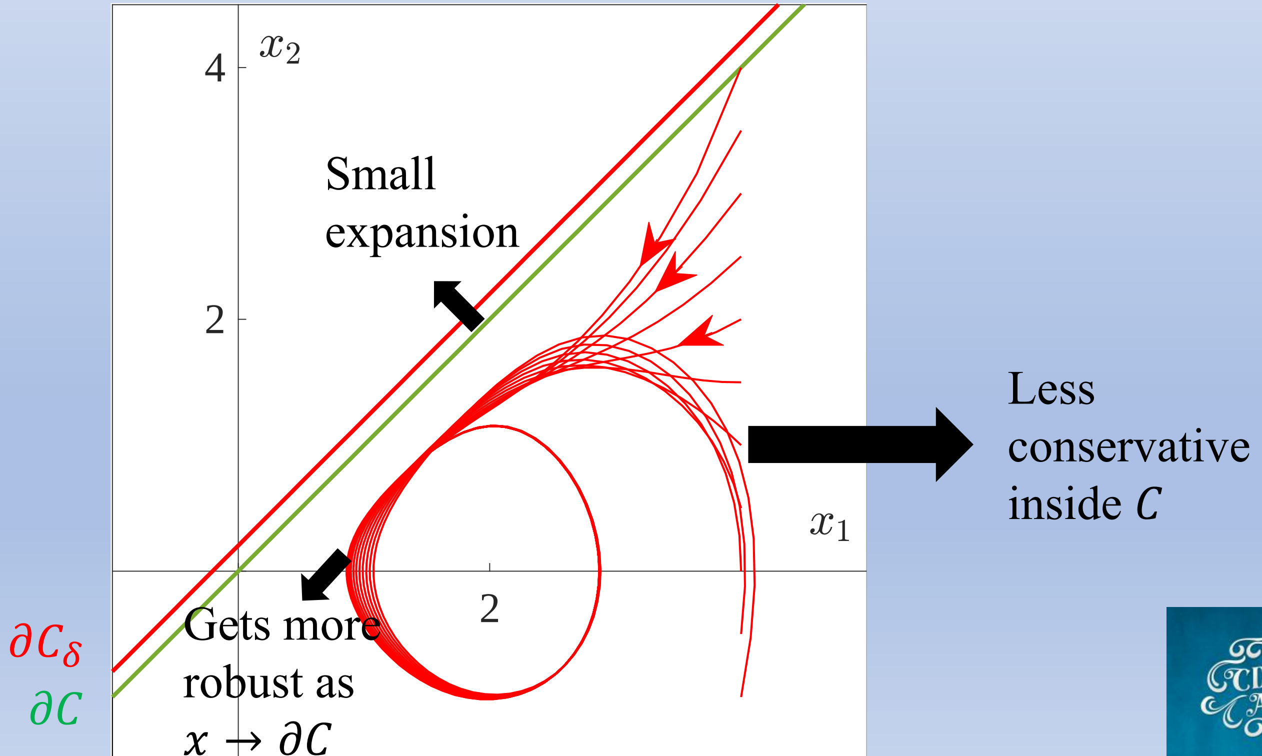
Example

Controller:

$$k_{\text{tissf}}(x) = k(x) - \frac{1}{\epsilon\big(h(x)\big)}$$

where $\epsilon(h(x)) = \epsilon_0 \mathrm{e}^{\lambda h(x)}$ with $\epsilon_0 > 0$ and $\lambda \geq 0$

can be shown $k_{\text{tissf}}(x) \in K_{\text{tissf}}(x)$



Results

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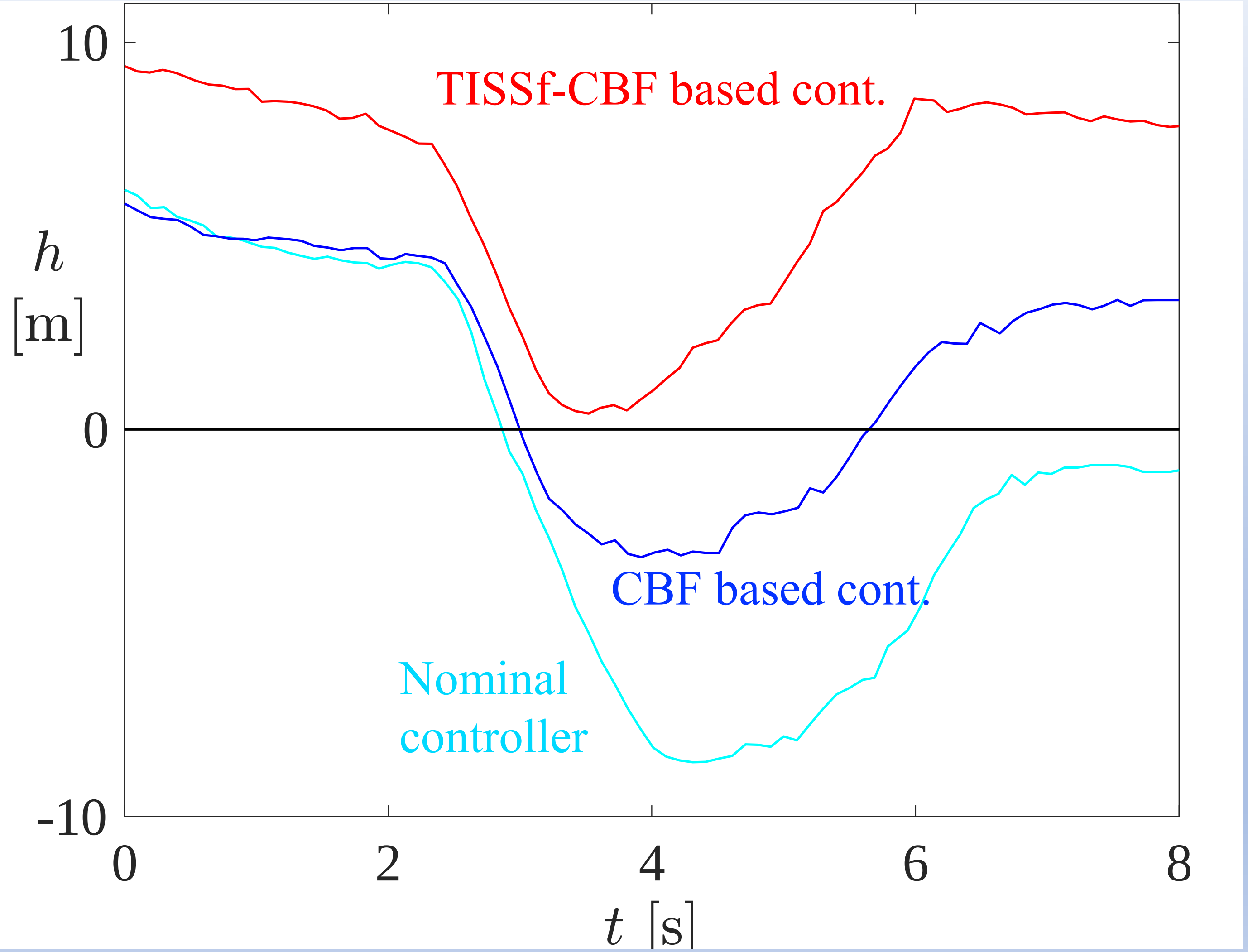

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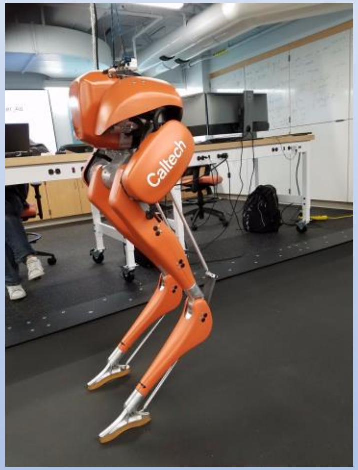
Future Work

- Determine ϵ effectively
  - With a user feedback: Preference-based learning[3]

- Incorporate with learning algorithms (such as RL or L-CBF[4]) to accommodate safe exploration in the presence of uncertainty.

Applications:

- Robotics



- Autonomous vehicles including lateral dynamics

[3] Tucker et. al, "Preference-Based Learning for Exoskeleton Gait Optimization", *IEEE International Conference on Robotics and Automation (ICRA)*, pp 2351-2357, 2020.
[4] Taylor et. al, "Learning for Safety-Critical Control with Control Barrier Functions", *Proceedings of Machine Learning Research,* vol 120, pp 1–10, 2020

Thank you for listening.
Questions?

Input to State Safety[2]

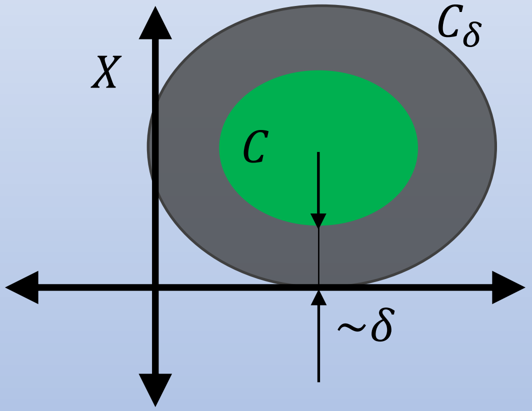
Input-to-State Safety (ISSf): If there exists a larger set C_δ that is forward invariant in the presence of the disturbance, then the system is *input-to-state safe*. The set C is ISSf set.

Let us consider

$$C_\delta = \{x \in X \mid h(x) + \gamma(\delta) \geq 0\}$$

Note $\gamma \in K_\infty$ → $C \subset C_\delta$ for $\delta > 0$

→ $C = C_\delta$ for $\delta = 0$



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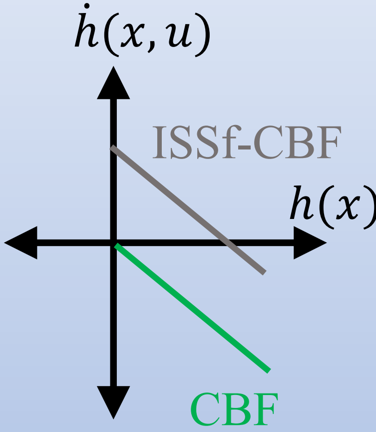

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Theorem[2]: Any controller $k(x) \in K_{\mathrm{issf}}(x)$ renders the set C_δ forward invariant (hence C ISSf set) for

$$\gamma(\delta) = -\alpha^{-1}\left(-\frac{\epsilon_0 \delta^2}{4}\right) \Longrightarrow \gamma(\delta) = \frac{\epsilon_0 \delta^2}{4\alpha_c} \quad \text{for } \alpha(r) = \alpha_c r$$

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[2] Kolathaya et. al, “Input-to-state safety with control barrier functions”, *IEEE Control Systems Letters*, Vol 3, pp 108–113, 2018.

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Truck Example

Longitudinal motion of a heavy-duty truck

- Preceding vehicle with emergency brake
- Both vehicle's GPS data available
- Simple (double integrator) model:

$$\dot{D} = v_\mathrm{L} - v$$
$$\dot{v} = u + d$$
$$\dot{v}_\mathrm{L} = a_\mathrm{L}$$

- A safe set C with a CBF:

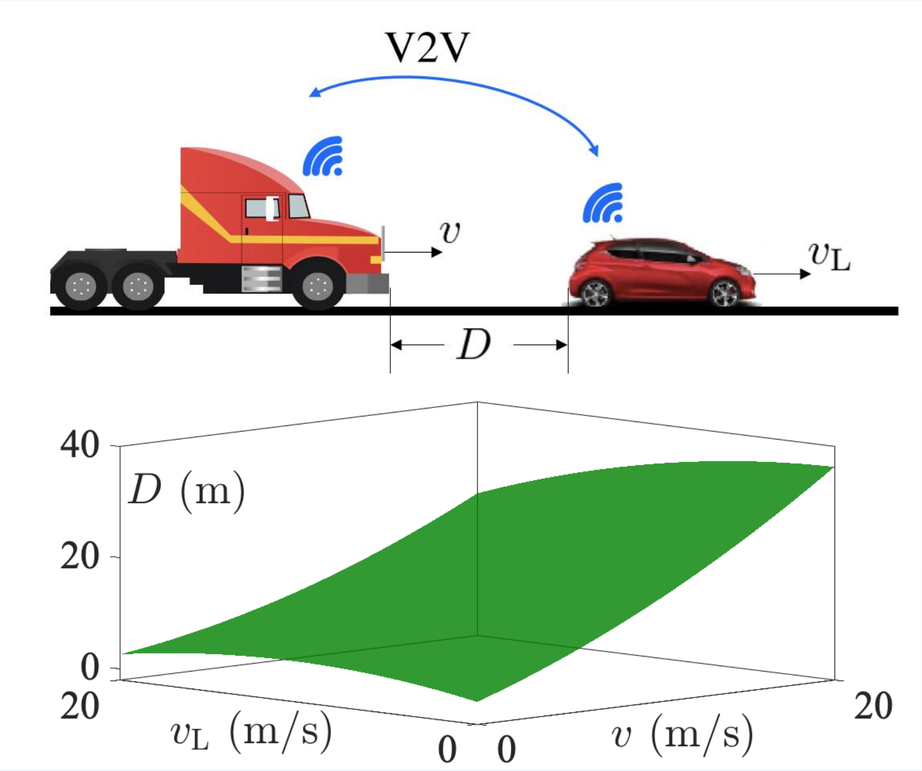
$$h(D, v, v_\mathrm{L}) = D - \widehat{D}(v, v_\mathrm{L})$$

- Quadratic Program based controller

$$k_\mathrm{QP}(D, v, v_\mathrm{L}) = \min\{k_\mathrm{n}(D, v, v_\mathrm{L}) \quad k_\mathrm{s}(D, v, v_\mathrm{L})\}$$

Nominal controller: OVM based Connected Cruise Control

Provably safe controller



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